MULTIPLIER AND ISOMORPHISM PROBLEMS OF SOME VECTOR VALUED FUNCTION SPACES

By

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MULTIPLIER AND ISOMORPHISM PROBLEMS OF SOME VECTOR VALUED FUNCTION SPACES

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CERTIFICATE

It is to certify that the work contained in the thesis entitled "Multiplier and isomorphism problems of some vector valued function spaces" by Mamta Kamra has been carried out under my supervision and has not been submitted elsewhere for a degree

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CHAPTER I

INTRODUCTION

In this thesis, we investigate multiplier and isomorphism problems of some Banach space or Banach algebra valued function spaces

Multiplier problems for various spaces of complex valued functions defined on a locally compact group G, have been studied by several authors. In particular, we mention Wendel [51], Brainerd and Edwards [2], Edwards [11,12] and Gaudry [14]. The appropriate choice of the definition of a multiplier depends on the context in which we are considering multipliers For various concepts and results regarding multipliers, one can refer to the book by Larsen [27].

Let $L^p(G)$ $(1 \leq p < \infty)$ denote the space of equivalence classes of complex valued measurable functions on G whose p^{th} power is integrable $C_o(G)$ denotes the space of complex valued continuous functions on G which vanish at infinity. M(G) denotes the space of bounded complex valued regular Borel measures on G. A bounded linear operator T on $L^1(G)$ is called a *multiplier* if $T(f * g) = f * Tg = (Tf) * g$ for all $f, g \in L^1(G)$. It is known [27] that a bounded linear operator T on $L^1(G)$ is a multiplier if and only if T commutes with translations. Wendel [51] gave a characterization of the multiplier space of $L^1(G)$. He proved that the multiplier space $M(L^1(G))$ is isometrically isomorphic to M(G). Brainerd and

Edwards [2] proved that the space $M(L^1(G), L^p(G))$, $1 < p < \infty$ can be identified with $L^p(G)$. He also proved that $M(C_o(G))$ is isometrically isomorphic to $M(G)$. Further, $C_o(G)$ is a Banach algebra with pointwise multiplication. Therefore, we can talk of pointwise multipliers of $C_o(G)$ It is easy to see that the space of pointwise multipliers of $C_o(G)$ can be identified with the space of bounded continuous functions on G We investigate these problems for the vector valued case.

In Chapter II, we collect basic definitions and facts about vector measures and vector valued function spaces which will be needed subsequently.

Let A be a Banach algebra and X, Y be Banach spaces. Lai [23] considered the problem of characterization of multipliers of $C_o(G,A)$ But there are a few gaps in his arguments. In Chapter III, we investigate these problems in more generality. We study $L^1(G,A)$-module homomorphisms of $C_o(G,A)$ We also note that if T is an $L^1(G,A)$-module homomorphism then T commutes with translations. But its converse need not be true. We first identify the left translation invariant operators from $C_o(G,X)$ to $C_o(G,Y)$ with $\mathcal{L}(X,Y^{**})$-valued measures satisfying certain conditions. Using this identification, we give a characterization of $L^1(G,A)$-module homomorphisms of $C_o(G,A)$. We also investigate the pointwise multipliers of $C_o(G,A)$ when G is a locally compact Hausdorff space

These problems were also investigated by Singh and Vasudeva [40] Our result and its proof regarding the pointwise multipliers of $C_o(G,A)$ is similar to that given by Singh and

Vasudeva Our approach to the investigation of $L^1(G,A)$-module homomorphisms is different from their approach and results also have different appearance.

Let G be a locally compact abelian group, A a commutative Banach algebra with identity and X an A-module. Tewari, Dutta and Vaidya [46] characterized the multiplier space of $L^1(G,A)$ Chan [4] obtained this result when G and A are arbitrary and A has a minimal approximate identity. Suppose $1 \leq p < \infty$. Lai [22] gave a characterization of the space $\mathrm{Hom}_{L^1(G,A)}(L^1(G,A), L^p(G,X))$ of $L^1(G,A)$-module homomorphisms from $L^1(G,A)$ to $L^p(G,X)$ (We shall frequently use the term multiplier in place of $L^1(G,A)$-module homomorphism). In Chapter IV, we prove a slightly better result. We give a characterization of the space

$\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A), L^p(G,X))$, when G and A are arbitrary and A has a minimal right approximate identity.

Let G be a locally compact abelian group and Γ its dual It is known [27] that a translation invariant operator T on $L^2(G)$ defines a bounded measurable function ϕ on Γ such that $(Tf)\hat{}(\gamma) = \phi(\gamma)\hat{f}(\gamma)$ for all $\gamma \in \Gamma$ and $f \in L^2(G)$. Conversely, every bounded measurable function on Γ defines a translation invariant operator on $L^2(G)$. We investigate this problem for the vector valued case We also note that in the scalar valued case, Plancherel theorem is the key step in proving this result But we know [37] that an analogue of Plancherel theorem is not true in general in the vector valued case. However, we prove that if G is a compact abelian group and H is a Hilbert space then the mapping $f \longrightarrow \hat{f}$ is

an isometry of $L^2(G,H)$ onto $\ell^2(\Gamma,H)$. Using this result, we give a characterization of translation invariant operators on $L^2(G,H)$, analogous to the scalar case. We also give an example to show that this result need not hold if H is not a Hilbert space.

Let G_1, G_2 be locally compact groups. Wendel [51] proved that if there exists a norm decreasing isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ then the groups G_1, G_2 are topologically isomorphic. Suppose $1 \leq p < \infty$. Gaudry [14] proved that the groups G_1, G_2 are topologically isomorphic if there exists an isometric isomorphism of $M(L^p(G_1))$ onto $M(L^p(G_2))$ To prove this result, a characterization of isometric multipliers of $L^p(G)$ plays a significant role Parrott [29] proved that isometric right multipliers of $L^p(G)$ are of the form $c\,\delta_t$ where c is scalar with $|c| = 1$.

In Chapter V, we investigate isometric isomorphism problems for the vector valued case. For this, we need to study isometric multipliers. Let G be a locally compact abelian group and A be a commutative semi-simple Banach algebra with a minimal approximate identity. We prove that surjective isometric multipliers of $L^1(G,A)$ are of the form $F\delta_t$ where F is a surjective isometric multiplier of A. In particular, if A has an identity then surjective isometric multipliers of $L^1(G,A)$ are of the form $a\,\delta_t$ where $a \in I(A) = \{b \in A: b \text{ is invertible, } \|b\| = \|b^{-1}\| = 1\}$. In case $1 < p < \infty$, we have characterized surjective isometric multipliers of $L^p(G,A)$ under certain conditions.

Let G_1, G_2 be locally compact abelian groups. Suppose A_1, A_2 are commutative semi-simple Banach algebras with identity such that they do not contain any two vectors x, y such that $\|\alpha x + \beta y\| = |\alpha| \|x\| + |\beta| \|y\|$ for all scalars α and β. In other words, $\ell_1^2 \notin A_1$. Note that every strictly convex space satisfies this condition Let $I(A_1)$ denote the set of isometric multipliers of A_1. Let $I'(A_1)$ be the closure of the linear hull of $I(A_1)$. We note that $I'(A_1)$ is a Banach algebra. We prove that if there is an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ then there exists a topological isomorphism of G_1 onto G_2 and an isometric isomorphism of $I'(A_1)$ onto $I'(A_2)$. In the case when $I'(A_1) = A_1$, we give a characterization of isometric isomorphisms of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$.

In the last chapter, we investigate bipositive isomorphism problems for Banach lattice algebra valued function spaces. The nature of problems discussed for bipositive case are similar to that for the isometric case.

Let G be a locally compact group and A be a Banach lattice algebra with identity. Let A^+ denote the set of all positive elements of A. A function $f \in L^1(G,A)$ is said to be *positive* if $f(s) \in A^+$ for almost all $s \in G$. An operator T on $L^1(G,A)$ is said to be *positive* if Tf is positive whenever f is a positive element of $L^1(G,A)$. T is said to be *bipositive* when Tf is positive if and only if f is positive. We prove that the positive multipliers of $L^1(G,A)$ are given by the measures belonging to $M(G,A)$ which take their values in A^+.

Let G_1, G_2 be locally compact groups. Let A_1, A_2 be Banach lattice algebras with identity such that inverse of each positive invertible element is positive. We prove that if there is a bipositive isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ then there exists a topological isomorphism of G_1 onto G_2 and a bipositive isomorphism of A_1 onto A_2 In the scalar valued case. this result was proved by Kawada [20]. Lastly, we give a characterization of bipositive isomorphisms of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$.

CHAPTER II

PRELIMINARIES

In this Chapter, we study some basic concepts of vector measures and vector valued function spaces, and collect some known results which will facilitate our later discussions.

§1. Vector Measures and Integration

For the definitions of vector measure, Bochner integral and other relevant concepts, the reader is referred to the books by Diestel and Uhl [6], Dinculeanu [7] and, Hille and Phillips [17].

Vector Measures

Let G be an arbitrary set and Σ be a σ-algebra of subsets of G A function μ from Σ to a Banach space X is called a *Vector measure* if whenever E_1 and E_2 are disjoint members of Σ then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

If $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ in the norm topology of X for all sequences $\{E_n\}$ of pairwise disjoint members of Σ then μ is called a *countably additive vector measure*.

The *variation* of μ is the extended non-negative function $|\mu|$ whose value on a set $E \in \Sigma$ is given by

$$|\mu|(E) = \sup_{\pi} \sum_{E \in \pi} \|\mu(E)\|,$$

where the supremum is taken over all partitions π of E into a

finite number of pairwise disjoint members of Σ.

If $|\mu|(G) < \infty$, then we say that μ is of *bounded variation*.

The *semivariation* of μ is the extended non-negative function $\|\mu\|$ whose value on a set $E \in \Sigma$ is given by

$$\|\mu\|(E) = \sup \left\{ |x^*\mu|(E): x^* \in X^*, \ \|x^*\| \leq 1 \right\},$$

where $|x^*\mu|$ is the variation of the scalar valued measure $x^*\mu$.
If $\|\mu\|(G) < \infty$ then μ is called a measure of *bounded semi-variation*.

Let $\mu : \Sigma \longrightarrow X$ be a countably additive vector measure and λ be a non-negative real valued measure on Σ. Then μ is called λ-*continuous* if

$$\lim_{\lambda(E) \to 0} \mu(E) = 0.$$

μ is λ-continuous if and only if μ vanishes on the sets of λ-measure zero [6, P.10].

Measurable functions

Let μ be a vector measure defined on (G, Σ). An X-valued function defined on G is called *simple* if it is of the form $\sum_{i=1}^{n} x_i \chi_{E_i}$ where $x_i \in X$ and χ_{E_i} is the characteristic function of E_i with $E_i \in \Sigma$.

A function $f : G \longrightarrow X$ is called μ-*measurable* if there exists a sequence of simple function $\{f_n\}$ such that $\lim_n \|f_n(s)-f(s)\| = 0$ $|\mu|$ - almost everywhere. f is called *weakly measurable* if the scalar valued function $x^*$of is μ-measurable for every $x^* \in X^*$.

The following Proposition correlates μ-measurability and weak μ-measurability.

Proposition 2.1.1 [6, P.42] : A function $f : G \longrightarrow X$ is μ-measurable if and only if

(i) f is weakly μ-measurable and

(ii) f is $|\mu|$-essentially separably valued i.e. there exists $E \in \Sigma$ with $|\mu|(E) = 0$ and such that $f(G\backslash E)$ is a norm separable subset of X.

Bochner Integral

Let X, Y, Z be Banach spaces We say that (X,Y,Z) forms a *bilinear system* if there exists a bilinear mapping $(x,y) \longrightarrow x\cdot y$ from $X \times Y$ into Z with $\|x\cdot y\|_Z \leq \|x\|_X \|y\|_Y$

Let μ be an Y-valued measure on (G, Σ) and f be an X-valued μ-measurable function defined on G. Then f is called *μ-integrable* if there exists a sequence of simple functions $\{f_n\}$ such that $\lim_{n,m} \int_G \|f_n - f_m\| \, d|\mu| = 0$. In this case, we define

$$\int_G f \, d\mu = \lim_n \int_G f_n \, d\mu$$

It is easy to see that if f is μ-integrable and $E \in \Sigma$ then $\chi_E f$ is μ-integrable and we define

$$\int_E f \, d\mu = \int_G \chi_E \, f \, d\mu.$$

Let $\mathbb{C}$ denote the field of complex numbers. We note that $(X, \mathbb{C}, X)$ forms a bilinar system if we define $x.\alpha = \alpha x$ for $x \in X$ and $\alpha \in \mathbb{C}$. This gives rise to the theory of integration of vector valued functions with respect to scalar valued measures (See §3.5 - §3.8 of [17] and Chapter II of [6]). In this case, we have the following proposition.

Proposition 2.1.2 [6, p. 47]: Let μ be a scalar valued measure and f be an X-valued μ-integrable function on G Let T be a bounded linear map from X to a Banach space Y. Then Tof is Y-valued μ-integrable function and

$$T\left(\int f\, d\mu\right) = \int (Tof)\, d\mu$$

Similarly $(\mathbb{C},X,X)$ forms a bilinear system with multiplication defined by $\alpha.x = \alpha x$ for $x \in X$ and $\alpha \in \mathbb{C}$. This leads us to the theory of integration of scalar valued functions with respect to vector measures. In this case, we have the following result.

Proposition 2.1.3: Let μ be an X-valued measure and f be a scalar valued μ - integrable function defined on G. Let T be a bounded linear map from X to a Banach space Y. Then $T\circ\mu$ is an Y-valued measure, f is $(T\circ\mu)$ - integrable and $T\left(\int f\, d\mu\right) = \int f\, d(T\circ\mu)$.

Lebesgue - Bochner Spaces

Let λ be a fixed positive measure on (G, Σ) and $1 \le p < \infty$. $L^p(G,\Sigma,\lambda,X)$ denotes the set of all equivalence classes of X-valued λ-measurable functions such that if f is a representative of an equivalence class belonging to $L^p(G,\Sigma,\lambda,X)$ then

$$\|f\|_p = \left(\int_G \|f(s)\|^p \, d\lambda(s) \right)^{1/p} < \infty.$$

$L^p(G,\Sigma,\lambda,X)$ becomes a Banach space with $\|.\|_p$.

$L^\infty(G,\Sigma,\lambda,X)$ denotes the set of all equivalence classes of essentially bounded, X-valued λ-measurable functions. For each representative f of an equivalence class belonging to $L^\infty(G,\Sigma,\lambda,X)$, define

$$\|f\|_\infty = \operatorname*{ess.sup}_s \|f(s)\|.$$

$L^{\infty}(G,\Sigma,\lambda,X)$ becomes a Banach space with $\| \ \|_{\infty}$.

If G is a locally compact group, Σ the σ - algebra of Borel subsets of G and λ the left Haar measure then we supress the letters Σ and λ and denote $L^{p}(G,\Sigma,\lambda,X)$ simply by $L^{p}(G,X)$.

It is known [8, p.227] that X-valued simple functions are dense in $L^{p}(G,\Sigma,\lambda,X)$ for $1 \leq p \leq \infty$ Therefore functions of the form $\sum_{i=1}^{n} x_i f_i$ where $x_i \in X$ and $f_i \in L^{p}(G,\Sigma,\lambda)$ are dense in $L^{p}(G,\Sigma,\lambda,X)$.

In the case when $X = \mathbb{C}$, $L^{p}(G,\Sigma,\lambda,X)$ is denoted by $L^{p}(G,\Sigma,\lambda)$.

§2. RNP and duality of $L^{p}(G,\Sigma,\lambda,X)$

Let λ be a finite positive measure on (G, Σ) A Banach space X is said to have Radon-Nikodym property with respect to λ if for each λ - continuous vector measure μ: $\Sigma \longrightarrow X$ of bounded variation there exists a $g \in L^{1}(G,\Sigma,\lambda,X)$ such that

$$\mu(E) = \int_{E} g \, d\lambda \text{ for all } E \in \Sigma.$$

A Banach space has *Radon-Nikodym property (RNP)* if X has RNP with respect to every finite measure space

Every reflexive space has RNP. Also any separable space which is the dual of another Banach space has RNP [6].

Let λ be a positive measure on (G, Σ). A Banach space X is said to have *wide RNP with respect to* λ if for each $K \in \Sigma$ with $\lambda(K) < \infty$, X has RNP with respect to λ_K where λ_K is defined as

$$\lambda_K(E) = \lambda(K \cap E) \text{ for all } E \in \Sigma \quad [24].$$

Note that for a finite measure space the concepts of RNP and wide RNP are the same.

The following result is due to Lai [24]. This result for a finite measure space is given in Diestel and Uhl [6].

Theorem 2.2.1: Let λ be a positive measure on (G, Σ), $1 < p < \infty$, p' the conjugate index of p and X a Banach space. Then $(L^p(G,\Sigma,\lambda,X))^*$ is isometrically isomorphic to $L^{p'}(G,\Sigma,\lambda,X^*)$ if and only if X^* has wide RNP with respect to λ. Further the correspondence between $U \in (L^p(G,\Sigma,\lambda,X))^*$ and $g \in L^{p'}(G,\Sigma,\lambda, X^*)$ is given by

$$U(f) = \int_G \langle f(s) , g(s) \rangle \, d\lambda(s) \text{ for all } f \in L^p(G,\Sigma,\lambda,X),$$

where $\langle , \rangle$ denotes the dual action.

In the case, when λ is a positive σ-finite measure on (G, Σ), the above correspondence holds for $p = 1$ also.

§3. Measures on a locally compact Hausdorff space

Throughout the section, G denotes a locally compact Hausdorff space, Σ the Borel σ-algebra of subsets of G and X a Banach space.

Regular Measures

Let μ be an X-valued measure defined on Σ. Then μ is said to be *regular* if for every $\epsilon > 0$, there exists a compact set K and an open set U with $K \subset E \subset U$ such that for any $F \in \Sigma$ with $F \subset U \backslash K$, we have $\|\mu(F)\| < \epsilon$.

For regular measures of bounded variation, we have the following result (See Props. 19 and 4 of § 15 of [18]).

Proposition 2.3.1: Let $\mu : \Sigma \longrightarrow X$ be a vector measure of bounded variation. Then μ is regular if and only if $|\mu|$ is regular.

The space of regular, X-valued measures on Σ of bounded variation is denoted by M(G,X). M(G,X) is a Banach space with the norm $\|\mu\| = |\mu|(G)$.

Let λ be a positive measure on (G, Σ) then for $f \in L^1(G,\Sigma,\lambda,X)$, define a mapping $\mu_f : \Sigma \longrightarrow X$ by

$$\mu_f(E) = \int_E f \, d\lambda.$$

It is easy to see that μ_f is an X-valued measure of bounded variation with $\|\mu_f\| = \|f\|_1$ Further, if λ is regular then μ_f is also regular. Hence in this case, $L^1(G,\Sigma,\lambda,X)$ is isometrically embedded in M(G,X).

$C_o(G,X)$ and its dual

Let $C_o(G,X)$ denote the space of all X-valued continuous functions on G vanishing at infinity. With the norm defined by

$$\|f\|_\infty = \sup_{s \in G} \|f(s)\|,$$

$C_o(G,X)$ becomes a Banach space. If X is a Banach algebra then $C_o(G,X)$ becomes a Banach algebra with pointwise multiplication.

It is known that functions of the form $\sum_{i=1}^{n} x_i f_i$ where $x_i \in X$ and $f_i \in C_o(G)$ are dense in $C_o(G,X)$.

In the case when $X = \mathbb{C}$, Riesz Representation theorem states that the dual of $C_o(G)$ is isometrically isomorphic to M(G). For the vector valued case, following result follows from §19 of [8].

Theorem 2.3.2 (Riesz Representation Theorem): $(C_o(G,X))^*$ is isometrically isomorphic to $M(G,X^*)$ under the correspondence $F \in (C_o(G,X))^*$ and $\mu \in M(G, X^*)$ defined by

$$F(f) = \int_G f \, d\mu \text{ for } f \in C_o(G, X).$$

For the compact space G, this result was proved by Singer [39].

Representation Theorems for bounded Linear Operators from $C_o(G)$ to a Banach Space

The following theorem gives a relationship between a bounded linear operator from $C_o(G)$ to X and an X^{**}- valued vector measure on Σ.

Theorem 2.3.3 [9, P.492]: If $U: C_o(G) \longrightarrow X$ is a bounded linear operator then there is a unique finitely additive vector measure $\mu: \Sigma \longrightarrow X^{**}$ such that

(i) the measure $\mu(.)x^*$ is in M(G) for every $x^* \in X^*$

(ii) the mapping $x^* \longrightarrow \mu(.)\, x^*$ from X^* to M(G) is weak* to weak* continuous

(iii) $Uf = \int f \, d\mu$ for every $f \in C_o(G)$

(iv) $\|U\| = \|\mu\|(G)$

Conversely, if $\mu: \Sigma \longrightarrow X^{**}$ is a vector measure satisfying (i) and (ii) then there is a bounded linear operator $U: C_o(G) \longrightarrow X$ which satisfies (iii) and (iv).

We say that the measure μ of Theorem 2.3.3 represents the operator U. The following Theorem gives a necessary and sufficient condition for the representing measure μ to take values in X.

Theorem 2.3.4 [9, p.493]: If μ: $\Sigma \longrightarrow X^{**}$ is a vector measure representing a bounded linear operator U: $C_o(G) \longrightarrow X$ then U is weakly compact if and only if μ takes its values in X In this case, μ is countably additive.

If U: $C_o(G) \longrightarrow X$ is a bounded linear operator then U is said to be *dominated* if for some non-negative $\lambda \in M(G)$,

$$\|Uf\| \leq \int |f| \, d\lambda \text{ for all } f \in C_o(G)$$

If U is dominated then the representing measure μ has bounded variation and $|\mu| \in M(G)$. Furthermore, dominated operators are weakly compact [52]. Consequently, if U: $C_o(G) \longrightarrow X$ is dominated then its representing measures μ belongs to M(G, X).

§4. Convolutions

Convolution of two measures

Let G be a locally compact group, λ the left Haar measure on G and Σ the σ-algebra of Borel subsets of G. Let X, Y, Z be Banach spaces such that (X,Y,Z) forms a bilinear system (See §2). Suppose $\mu \in M(G,X)$ and $\nu \in M(G,Y)$. Then $\mu * \nu$ is a Z-valued measure defined by [7, §24]

$$(\mu * \nu)(f) = \int_G \int_G f(st) \, d\mu(s) \, d\nu(t) \text{ for all } f \in C_o(G).$$

Further,

$$\|(\mu * \nu)(f)\| \leq \int_G \int_G |f(st)| \, d|\mu|(s) \, d|\nu|(t)$$

$$= (|\mu| * |\nu|)(|f|).$$

Since μ and ν are regular measures, it follows that $|\mu| * |\nu|$ is a regular measure. It follows by comments subsequent to Theorem 2.3.4 that $\mu * \nu$ belongs to M(G,Z).

If A is a Banach algebra then M(G,A) becomes a Banach algebra with multiplication defined by convolution.

Convolution of a measure and a function

If $f \in L^1(G,X)$ and $\mu \in M(G,Y)$ then the integral

$(\mu * f)(s) = \int f(t^{-1}s)\, d\mu(t)$ exists for almost all $s \in G$.

Further, $\mu * f \in L^1(G,Z)$ and $\|\mu * f\|_1 \leq \|\mu\|\, \|f\|_1$

Let Δ denote the modular function. Then the integral

$$(f * \mu)(s) = \int f(st^{-1})\, \Delta(t^{-1})\, d\mu(t)$$

exists for almost all $s \in G$ We also have $f * \mu \in L^1(G,Z)$ and $\|f * \mu\|_1 \leq \|f\|_1\, \|\mu\|$ [7, §24].

If A is a Banach algebra then $L^1(G,A)$ is a two-sided ideal of M(G,A).

Convolution of two functions

If $f \in L^1(G,X)$ and $g \in L^p(G,Y)$, $1 \leq p \leq \infty$ then the integral

$$(f * g)(s) = \int f(t)\, g(t^{-1}s)\, d\lambda(t) = \int f(st)\, g(t^{-1})\, d\lambda(t)$$

exists almost everywhere.

Further $f * g \in L^p(G,Z)$ and $\|f * g\|_p \leq \|f\|_1\, \|g\|_p$. We also have,

$$(f * g)(s) = \begin{cases} \int f(t^{-1})\, g(ts)\ \ \Delta(t^{-1})\, d\lambda(t) \\ \int f(st^{-1})\, g(t)\ \ \Delta(t^{-1})\, d\lambda(t) \end{cases}$$

If A is a Banach algebra and X is an A-module then $L^p(G,X)$ becomes a left $L^1(G,A)$-module with multiplication defined by convolution.

If $f \in L^1(G,X)$ and $g \in C_0(G,Y)$ then $f * g$ is defined analogously. It is known that $f * g$ exists everywhere and belongs to $C_0(G,Z)$ with $\|g * f\|_\infty \leq \|g\|_\infty\, \|f\|_1$.

If A is a Banach algebra and X is an A-module then $C_o(G,X)$ is a left $L^1(G,A)$ - module.

If G is unimodular then $L^p(G,X)$ and $C_o(G,X)$ are also right $L^1(G,A)$-modules [7, §24].

§5. The Algebra $L^1(G,A)$

We have noted in the last section that if A is a Banach algebra and G is a locally compact group then $L^1(G,A)$ becomes a Banach algebra with convolution as multiplication Furthermore, if G and A are commutative then $L^1(G,A)$ is a commutative Banach algebra.

The following results are well known.

1) $L^1(G,A)$ is semi-simple if and only if A is semi-simple [19].
2) $L^1(G,A)$ is regular if and only if A is regular [48].
3) $L^1(G,A)$ is tauberian if A is tauberian [15]
4) $L^1(G,A)$ has a bounded approximate identity if and only if A has a bounded approximate identity [10]

The following result about the maximal ideal space of $L^1(G,A)$ is due to Johnson [19].

Theorem 2.5.1: Let G be a locally compact abelian group and Γ its dual. Suppose A is a commutative Banach algebra Then the maximal ideal space of $L^1(G,A)$ is homeomorphic to $\Gamma \times \Delta(A)$ where $\Delta(A)$ denotes the maximal ideal space of A. Further the Gelfand transform $\mathcal{F}f$ of $f \in L^1(G,A)$ is given by

$$(\mathcal{F}f)(\gamma,\phi) = \int_G \phi(f(s))\ \overline{\gamma(s)}\ d\lambda(s) \text{ for } (\gamma,\phi) \in \Gamma \times \Delta(A).$$

We define the Fourier transform of $f \in L^1(G,A)$ by

$$\hat{f}(\gamma) = \int_G f(s)\ \overline{\gamma(s)}\ d\lambda(s) \text{ for } \lambda \in \Gamma.$$

Note that $\hat{f}$ defines a function on Γ to A We have the following relationship between the Gelfand transform and the Fourier transform.

$$(\mathcal{F}f)(\gamma,\phi) = \phi\ (\hat{f}(\gamma)) \text{ for } (\gamma,\phi) \in \Gamma \times \Delta\ (A).$$

§6. Multipliers

Let f be a function defined on a locally compact group G. For $s \in G$, we define $(\tau_s f)(t) = f(s^{-1}t)$ for $t \in G$. $\tau_s f$ is called the *left translate* of f The *right translate* $\tau^s f$ of f is defined by $\tau^s f\ (t) = f(ts^{-1})$ for $t \in G$.

Definition 2.6.1: Let B be a space of functions defined on a locally compact group G. Then B is called *translation invariant* if both $\tau_s f$ and $\tau^s f$ belong to B for all $f \in B$ and $s \in G$.

Definition 2.6.2 Let B_1 and B_2 be two translation invariant spaces of functions defined on a locally compact group G. An operator T: $B_1 \longrightarrow B_2$ is said to be *left translation invariant* if $T(\tau_s f) = \tau_s(Tf)$ for all $f \in B_1$ and $s \in G$ i.e. T commutes with left translations [27].

Similarly we define the right translation invariant operators from B_1 to B_2. An operator from B_1 to B_2 is said to be *translation invariant* if it is both left and right translation invariant.

Definition 2.6.3: Let X, Y be left A-modules. A continuous linear operator T: $X \longrightarrow Y$ is said to be a *left A-module homomorphism* if $T(ax) = a\ (Tx)$ for all $a \in A$ and $x \in X$ [25,40].

The set of all left A-module homomorphisms from X to Y is a Banach space under the operator norm and is denoted by $\mathrm{Hom}_A^{\ell}(X,Y)$. If X = Y then $\mathrm{Hom}_A^{\ell}(X,Y)$ is denoted by $\mathrm{Hom}_A^{\ell}(X)$

If X and Y are right A-modules then the right A-module homomorphisms from X to Y are analogously defined. We denote the space of all right A-module homomorphisms from X to Y by $\mathrm{Hom}_A^{r}(X,Y)$.

If X and Y are A-modules then $\mathrm{Hom}_A(X,Y)$ denotes the space of all continuous linear operators from X to Y which are both left and right A-module homomorphisms.

Definition 2.6.4: Let A be a Banach algebra. A continuous linear operator T on A is called a *left multiplier* if T(xy) = (Tx)y for all $x,y \in A$ [27].

The set of all left multipliers of A is a Banach algebra under the operator norm and is denoted by $M^{\ell}(A)$. The right multipliers of A are analogously defined $M^{r}(A)$ denotes the space of all right multipliers of A

A continuous linear operator on A which is both left and right multiplier is called a *multiplier* of A. The space of all multipliers of A is denoted by M(A) Observe that if X = Y = A then $\mathrm{Hom}_A(X,Y) = M(A)$.

We have noted in section 4 that if X is an A-module then $C_o(G,X)$ and $L^p(G,X)$ for $1 \le p < \infty$ are left $L^1(G,A)$ - modules with module action defined by convolution. Also for a locally compact Hausdorff space G, $C_o(G,A)$ becomes a Banach algebra with pointwise multiplication.

In view of the above discussion, we can talk of the spaces $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$, $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A), L^p(G,X))$ and $M^{\ell}(C_o(G,A))$.

If $X = A = \mathbb{C}$ then these spaces are denoted by $\mathrm{Hom}^{\ell}_{L^1(G)}(C_o(G))$, $\mathrm{Hom}^{\ell}_{L^1(G)}(L^1(G))$, $L^p(G))$ and $M^{\ell}(C_o(G))$ respectively.

§7. Miscellaneous

Banach Alaglou's Theorem 2 7.1: [9, V 4 2]: Let X be a Banach space. Then any closed norm bounded ball in X^* is compact in weak* topology

Fubini's theorem 2.7 2 [16, 13.8] Let G_1, G_2 be locally compact Hausdorff spaces and let ν_1, ν_2 be regular non-negative Borel measures on G_1, G_2 respectively. Let f be in $L^1(G_1 \times G_2 \quad \nu_1 \times \nu_2)$. Then the function $s \longrightarrow f(s,t)$ belongs to $L^1(G_1,\nu_1)$ for ν_2 - almost all $t \in G_2$ and the function $t \longrightarrow f(s,t)$ belongs to $L^1(G_2,\nu_2)$ for ν_1-almost all $s \in G_1$.

Furthermore, the function defined by $s \longrightarrow \int_G f(s,t)\, d\nu_2(t)$ where the integral exists and o elsewhere, is in $L^1(G_1,\nu_1)$, similarly for $\int_{G_1} f(s,t)\, d\nu_1(s)$. Finally, we have

$$\int_{G_1 \times G_2} f(s,t)\, d(\nu_1 \times \nu_2)(s,t) = \begin{cases} \int_{G_1}\int_{G_2} f(s,t)\, d\nu_2(t)\, d\nu_1(s) \\ \int_{G_1}\int_{G_2} f(s,t)\, d\nu_1(t)\, d\nu_2(s) \end{cases}$$

Factorization theorem 2.7.3 [16, 32 .2 2]: Let A be a Banach algebra with a bounded left approximate identity and X be a left Banach A-module. Then for each $z \in X$ and $\epsilon > 0$, there exist elements $a \in A$ and $y \in X$ such that

(i) $z = a.y$

(ii) $y \in (A.z)^-$

(iii) $\|y-z\| \leq \epsilon$

CHAPTER III

MULTIPLIERS OF $C_0(G,A)$

§1. INTRODUCTION

Let G be a locally compact group and A be a Banach algebra. We have already observed in Chapter II that $C_o(G,A)$ is a left $L^1(G,A)$-module. Therefore we can talk of left $L^1(G,A)$-module homomorphisms of $C_o(G,A)$. If however, G is unimodular then $C_o(G,A)$ is also a right $L^1(G,A)$-module. Therefore in this case, it is also meaningful to talk of right $L^1(G,A)$-module homomorphisms of $C_o(G,A)$.

Lai [23] had investigated the space $\mathrm{Hom}_{L^1(G,A)}(C_o(G,A))$ of $L^1(G,A)$-module homomorphisms of $C_o(G,A)$ under the hypothesis that G is a locally compact abelian group and A is a commutative Banach algebra with identity However, his results need some modifications. The main problem with his arguments lies in the identification of the space of continuous linear functions from $C_o(G,A)$ to A with the space M(G,A). This does not seem to be correct. The space has already been characterized as $\mathcal{L}(A,A^{**})$-valued measures satisfying certain conditions, whereas the space of dominated continuous linear functions from $C_o(G,A)$ to A is identified with $M(G,\mathcal{L}(A))$ (See [8]). Furthermore, even for commutative Banach algebras A with identity of dimension > 1, $A \cong M(A) \subsetneq \mathcal{L}(A)$. Therefore each continuous linear operator from

$C_o(G,A)$ to A need not arise from a measure belonging to $M(G,A)$.

In section 2, we shall use the characterization of A-valued dual of $C_o(G,A)$ to give a description of the space $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$ when the algebra A has a bounded right approximate identity.

Lai [23] had also investigated the pointwise multipliers of $C_o(G,A)$ He claims that $M(C_o(G,A))$ is isometrically isomorphic to the space $C^b(G,M(A))$ of bounded functions from G to M(A) continuous in the uniform operator topology However, there appears to be a slight error in his arguments. It turns out that $M(C_o(G,A))$ is isometrically isomorphic to the space $C^b_s(G,M(A))$ of bounded functions from G to M(A) continuous in the strong operator topology. Furthermore, in this case the group structure of G does not play any role and therefore G may be assumed to be a locally compact Housdorff space. In Section 3, we characterize the space $M^{\ell}(C_o(G,A))$ under the hypothesis that A is a Banach algebra with a right approximate identity.

When we were writing this thesis, we came across the paper by Singh and Vasudeva [40] They have also noted the problems with the arguments of Lai in [23] and have given characterizations of $\mathrm{Hom}^{r}_{L(G,A)}(C_o(G,A))$ and $M^{\ell}(C_o(G,A))$. Our proof regarding the pointwise multipliers is more or less same as the proof given in [40]. Our approach to the investigation of $L^1(G,A)$-module homomorphisms of $C_o(G,A)$ differs from that in [40] and the results also have a different appearance. The main difference arises due to the different approaches to the concept of vector measures. Our approach is that of Diestel [6] and Dinculeanu [8] whereas

the approach in [40] is that of Dinculeanu [7].

§2. $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$

Throughout this section, G denotes a locally compact group and A a Banach algebra with a bounded right approximate identity unless otherwise stated.

In the scalar valued case, it is known that $T \in \mathrm{Hom}_{L^1(G)}(C_o(G))$ if and only if T commutes with translations. This need not be true in the vector valued case. If $T \in \mathrm{HOm}_{L^1(G,A)}(C_o(G,A))$ then T commutes with translations. However, even if A is a commutative Banach algebra with identity of dimension > 1, one can construct examples as in [46] to show that a translation invariant operator on $C_o(G,A)$ need not be an $L^1(G,A)$-module homomorphism. Thus the problem of characterization of translation invariant operators becomes important. In this section, we characterize the left translation invariant operators from $C_o(G,X)$ to $C_o(G,Y)$ where X,Y are Banach spaces. We use this result to give a characterization of the space $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$.

For the characterization of left translation invariant operators from $C_o(G,X)$ to $C_o(G,Y)$, we shall need a representation theorem for continuous linear operators from $C_o(G,X)$ to Y due to Dinculeanu [8].

3.2.1 : Let Σ denote the σ-algebra of Borel subsets of G. Let $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ be a vector measure. For each $y^* \in Y^*$ and $E \in \Sigma$, define

$$\mu_{y^*}(E)(x) = y^*\left(\mu(E)(x)\right) \text{ for all } x \in X.$$

Then $\mu_{y^*} : \Sigma \longrightarrow X^*$ is a vector measure.

For each $E \in \Sigma$, define

$$\tilde{\mu}(E) = \sup\left\{|\mu_{y^*}|(E) : \|y^*\| \leq 1\right\}$$

$\tilde{\mu}(E)$ is called the *semi-variation* of μ of E. If $\tilde{\mu}(G) < \infty$, we say that μ has finite semi-variation.

We now state the representation theorem for continuous linear operators from $C_o(G,X)$ to Y Its proof follows from §19 of [8] (see also Theorem 1, P. 152 of [6]).

Theorem 3.2.2 : Suppose $U : C_o(G,X) \longrightarrow Y$ is a bounded linear operator. Then there exists a unique vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ such that

(i) μ has finite semi-variation.

(ii) For each $y^* \in Y^*$, $\mu_{y^*} \in M(G,X^*)$.

(iii) The mapping $y^* \longrightarrow \mu_{y^*}$ is weak* to weak* continuous from Y^* to $(C_o(G,X))^*$.

(iv) $Uf = \int_G f \, d\mu$ for all $f \in C_o(G,X)$.

(v) $\|U\| = \tilde{\mu}(G)$.

Conversely, any vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ which satisfies (i), (ii) and (iii) defines a bounded linear operator $U : C_o(G,X) \longrightarrow Y$ by means of (iv) such that (v) holds.

The following result shows that if μ satisfies the conditions of Theorem 3.2.2 then $\mu * f$ is defined for all $f \in C_o(G,X)$ as a

continuous function.

Proposition 3.2.3 : Let $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ be a vector measure of finite semi-variation such that for each y^*, $\mu_{y^*} \in M(G,X^*)$ and the mapping $y^* \longrightarrow \mu_{y^*}$ is weak* to weak* continuous from Y^* to $(C_o(G,X))^*$. Then for $f \in C_o(G,X)$ and $s \in G$,
$(\mu * f)(s) = \int_G f(y^{-1}s)\, d\mu(y)$ exists as an element of Y and $\mu * f$ is a continuous function on G.

Proof: Let $f \in C_o(G,X)$ and $s \in G$ Then $\tau_s \tilde{f} \in C_o(G,X)$ and

$$(\mu * f)(s) = \int_G f(y^{-1}s)\, d\mu(y)$$

$$= \int_G (\tau_s \tilde{f})(y)\, d\mu(y).$$

Therefore it follows from Theorem 3.2.2 that $(\mu * f)(s)$ exists as an element of Y

We shall show that $\mu * f$ is a continuous function.

Let $s,t \in G$. Then

$$\|(\mu * f)(s) - (\mu * f)(t)\| = \left\| \int_G f(y^{-1}s)\, d\mu(y) - \int_G f(y^{-1}t)\, d\mu(y) \right\|$$

$$= \left\| \int_G \left[(\tau_s \tilde{f})(y) - (\tau_t \tilde{f})(y) \right] d\mu(y) \right\|$$

$$= \sup_{\|y^*\| \le 1} \left| y^* \left(\int_G (\tau_s \tilde{f} - \tau_t \tilde{f})(y)\, d\mu(y) \right) \right|$$

$$= \sup_{\|y^*\| \le 1} \left| \int_G (\tau_s \tilde{f} - \tau_t \tilde{f})(y)\, d\mu_{y^*}(y) \right|$$

$$\le \sup_{\|y^*\| \le 1} \|\tau_s \tilde{f} - \tau_t \tilde{f}\|_\infty \, |\mu_{y^*}|(G)$$

$$= \|\tau_s \tilde{f} - \tau_t \tilde{f}\|_\infty \, \tilde{\mu}(G),$$

which tends to zero as $s \longrightarrow t$. Therefore $\mu * f$ is continuous.

In the following Theorem, we give a characterization of left translation invariant operators from $C_o(G,X)$ to $C_o(G,Y)$ which is a key step for the characterization of the space $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$.

Theorem 3.2.4 : Let X,Y be Banach spaces. Suppose $T : C_o(G,X) \longrightarrow C_o(G,Y)$ is a continuous linear left translation invariant operator. Then there exists a unique vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ such that

(i) μ has finite semi-variation.

(ii) $\mu_{y^*} \in M(G,X^*)$ for each $y^* \in Y^*$.

(iii) The mapping $y^* \longrightarrow \mu_{y^*}$ is weak* to weak* continuous from Y^* to $(C_o(G,X))^*$.

(iv) $Tf = \mu * f$ for all $f \in C_o(G,X)$.

(v) $\|T\| = \tilde{\mu}(G)$.

(vi) For each $f \in C_o(G,X)$, the mapping $s \longrightarrow \int \tau_s f \, d\mu$ from G to Y vanishes at infinity.

Conversely, if $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ is a vector measure of finite semi-variation such that (ii), (iii) and (vi) hold then T defined by (iv) is a continuous linear left translation invariant operator from $C_o(G,X)$ to $C_o(G,Y)$ such that (v) holds.

Proof : Let e denote the identity of the group G. We define $U : C_o(G,X) \longrightarrow Y$ by

$$Uf = (\tilde{Tf})(e) \text{ for all } f \in C_o(G,X).$$

Then U is a continuous linear mapping. By Theorem 3.2.2, there exists a unique vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(X,Y^{**})$ such that for each $f \in C_o(G,X)$, we have

$$Uf = \int f \, d\mu.$$

Since $Uf = (\tilde{Tf})(e)$, we get that

$$(\tilde{Tf})(e) = (\mu * \tilde{f})(e) \ .$$

Therefore for each $f \in C_o(G,X)$,

$$(Tf)(e) = (\mu * f)(e)$$

Since T commutes with left translations, for $s \in G$ we have

$$\begin{aligned}(Tf)(s) &= (\tau_{s^{-1}}(Tf))(e) \\ &= (T(\tau_{s^{-1}}f))(e) \\ &= (\mu * \tau_{s^{-1}}f)(e) \\ &= \tau_{s^{-1}}(\mu * f)(e) \\ &= (\mu * f)(s).\end{aligned}$$

Thus we have $Tf = \mu * f$ for each $f \in C_o(G,X)$

Furthermore, $(\tilde{Tf})(s) = (\mu * \tilde{f})(s)$

$$\begin{aligned}&= \int \tilde{f}(y^{-1}s) \, d\mu(y) \\ &= \int f(s^{-1}y) \, d\mu(y) \\ &= \int \tau_s f \, d\mu.\end{aligned}$$

Since $\tilde{Tf} \in C_o(G,Y)$, it implies that for each $f \in C_o(G,X)$ the mapping $s \longrightarrow \int \tau_s \, d\mu$ from G to Y vanishes at infinity.

Since $Uf = (\tilde{Tf})(e)$, we get that

$$\|Uf\| = \|(\tilde{Tf})(e)\| \leq \|\tilde{Tf}\|_\infty \leq \|T\| \, \|f\|_\infty .$$

Thus $\|U\| \leq \|T\|$. (2.1)

Furthermore, since T commutes with left translations

$$\|T\| = \sup_{\|f\|_\infty \leq 1} \|Tf\|_\infty$$

$$= \sup_{\|f\|_\infty \leq 1} \sup_{s \in G} \|(Tf)(s)\|$$

$$= \sup_{\|f\|_\infty \leq 1} \sup_{s \in G} \|(\tau_{s^{-1}}(Tf))(e)\|$$

$$= \sup_{\|f\|_\infty \leq 1} \sup_{s \in G} \|(T(\tau_{s^{-1}}f))(e)\|$$

$$= \sup_{\|f\|_\infty \leq 1} \sup_{s \in G} \|U(\tau_{s^{-1}}\tilde{f})\| \leq \|U\|$$

Therefore $\|T\| \leq \|U\|$. (2.2)

Combining (2.1) and (2.2), we get $\|T\| = \|U\|$

By Theorem 3.2.2, $\|U\| = \tilde{\mu}(G)$.

Therefore $\|T\| = \tilde{\mu}(G)$.

Thus the vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(X, Y^{**})$ satisfies all the conditions (i) - (vi).

Conversely, suppose $\mu : \Sigma \longrightarrow \mathcal{L}(X, Y^{**})$ is a vector measure of finite semi-variation which satisfies the conditions (ii), (iii) and (vi). Then for $f \in C_o(G,X)$, we define $Tf = \mu * f$. By Proposition 3.2.3, for each $s \in G$, $(\mu * f)(s)$ exists as an element of Y and $\mu * f$ is a continuous function on G. We shall show that $\mu * f$ vanishes at infinity.

Since $(\tilde{Tf})(s) = \int \tau_s f \, d\mu$, therefore it follows by (vi) that $(\tilde{Tf}) \in C_o(G,Y)$ for all $f \in C_o(G,X)$. In particular $Tf = \mu * f \in$

$C_o(G,Y)$ for each $f \in C_o(G,X)$

It is easy to see that T commutes with left translations and

$$\|T\| = \tilde{\mu}(G).$$

Thus T defines a continuous linear map from $C_o(G,X)$ to $C_o(G,Y)$ which commutes with left translations.

This completes the proof of the theorem

Remark 3.2.5 : We can analogously characterize the continuous linear right translation invariant operators on $C_o(G,A)$.

As pointed out in the beginning of this section that a translation invariant operator on $C_o(G,A)$ need not be an element of $\mathrm{Hom}_{L^1(G,A)}(C_o(G,A))$. In the following Proposition, we give a necessary and sufficient condition for a continuous linear operator on $C_o(G,A)$ to belong to $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$.

Proposition 3.2.6 : Let G be a locally compact group and A be a Banach algebra with a bounded right approximate identity. Let T be a continuous linear operator on $C_o(G,A)$. Then the following are equivalent.

(i) $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$.

(ii) $T\tau_s = \tau_s T$ and $T(af) = a(Tf)$ for all $f \in C_o(G,A)$, $a \in A$ and $s \in G$.

Proof : Suppose (i) holds. Since the linear span of $\{bg : b \in A, g \in C_o(G)\}$ is dense in $C_o(G,A)$, it is enough to show that

$$T(\tau_s\, bg) = \tau_s(T(bg)) \text{ and } T(abg) = a(T(bg)) \text{ for all } s \in G,$$

$a, b \in A$ and $g \in C_o(G)$.

By Hewitt-Factorization theorem [16, P 270] there exist elements b_1, $b_2 \in A$ such that $b = b_1 b_2$.

Also we know that $L^1(G) * C_o(G) = C_o(G)$. Therefore there exist function $g_1 \in L^1(G)$ and $g_2 \in C_o(G)$ such that $g = g_1 * g_2$.

Now for $s \in G$,

$$\begin{aligned}
\tau_s(T(bg)) &= \tau_s(T(b_1 b_2\, g_1 * g_2)) \\
&= \tau_s(T((b_1 g_1) * (b_2 g_2))) \\
&= \tau_s((b_1 g_1) * T(b_2 g_2)) \\
&= (b_1(\tau_s g_1)) * T(b_2 g_2) \\
&= T(b_1(\tau_s g_1) * b_2 g_2) \\
&= T(\tau_s(bg)).
\end{aligned}$$

Thus, we get that $\tau_s\, T = T\, \tau_s$

Furthermore,

$$\begin{aligned}
T(abg) &= T(a\, b_1 b_2\, g_1 * g_2) \\
&= T(a b_1 g_1 * b_2 g_2) \\
&= (a b_1 g_1) * (T(b_2 g_2)) \\
&= a(b_1 g_1) * (T(b_2 g_2)) \\
&= a((b_1 g_1) * (T(b_2 g_2))) \\
&= a(T(b_1 g_1 * b_2 g_2)) \\
&= a(T(bg)).
\end{aligned}$$

This proves (ii).

Now suppose that (ii) holds. Since the linear span of $\{bg : b \in A,\ g \in L^1(G)\}$ is dense in $L^1(G,A)$, it is enough to prove that $T(bg * f) = bg * Tf$ for all $b \in A$, $g \in L^1(G)$ and $f \in C_o(G,A)$.

Clearly,

$$T(bg * f) = T(b(g * f))$$

$$= b(T(g * f))$$

Also, $$T(g * f) = T\left(\int g(s)\, \tau_s f \, ds \right)$$

$$= \int g(s)\, T(\tau_s f)\, ds$$

$$= \int g(s)\, \tau_s(Tf)\, ds$$

$$= g * Tf$$

Therefore, $T(bg * f) = b(g * Tf)$

$$= bg * Tf.$$

This proves (1) and completes the proof of the Proposition.

We now use Theorem 3.2.4 and Proposition 3.2.6 to give a characterization of the space $\mathrm{Hom}^{\ell}_{L^1(G,A)}(C_0(G,A))$.

3.2.7 : Let A be a Banach algebra and X be an A-module. Then X^* and X^{**} are also A-modules with module action defined as follows [22].

Let $a \in A$, $x \in X$ and $x^* \in X^*$. We define

$$(a \cdot x^*)(x) = x^*(xa) \text{ and } (x^* \cdot a)(x) = x^*(ax)$$

Let $x^{**} \in A^{**}$. We define

$$(a \cdot x^{**})(x^*) = x^{**}(x^* \cdot a) \text{ and } (x^{**} \cdot a)(x^*) = x^{**}(a \cdot x^*).$$

In particular, A^* and A^{**} are A-modules.

Theorem 3.2.8 : Let G be a locally compact group and A be a Banach algebra with a bounded right approximate identity. Let $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(C_0(G,A))$. Then there exists a unique vector measure $\mu : \Sigma \longrightarrow \mathrm{Hom}^{\ell}_{A}(A, A^{**})$ such that

(1) μ is of finite semi-variation.

(ii) For each $a^* \in A^*$, $\mu_{a^*} \in M(G,A^*)$

(iii) The mapping $a^* \longrightarrow \mu_{a^*}$ is weak* to weak* continuous from A^* to $(C_o(G,A))^*$

(iv) $Tf = \mu * f$ for each $f \in C_o(G,A)$

(v) $\|T\| = \tilde{\mu}(G)$.

(vi) For each $f \in C_o(G,A)$, the mapping $s \longrightarrow \int \tau_s f \, d\mu$ from G to A vanishes at infinity.

Conversely, suppose $\mu : \Sigma \longrightarrow \mathrm{Hom}_A^{\ell}(A,A^{**})$ is a vector measure of finite semi-variation which satisfies (ii), (iii) and (vi). Then μ defines an element $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$ by means of (iv) such that (v) holds.

Proof : Suppose $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$. Let $f \in C_o(G,A)$, $a \in A$ and $s \in G$. Then by Proposition 3.2.6, we have

$$T\tau_s = \tau_s T \text{ and } T(af) = a(Tf).$$

By Theorem 3.2.4, there exists a unique vector measure $\mu : \Sigma \longrightarrow \mathcal{L}(A,A^{**})$ which satisfies all the conditions (i) to (vi). We need only check that μ takes values in $\mathrm{Hom}_A^{\ell}(A,A^{**})$.

Since $(\tilde{Tf})(e) = \int_G f \, d\mu$, therefore for $a^* \in A^*$,

$$a^*\left((\tilde{Tf})(e)\right) = a^*\left(\int_G f \, d\mu\right)$$

$$= \int_G f \, d\mu_{a^*} .$$

In particular, take $f = abg$ where $a,b \in A$ and $g \in C_o(G)$. Then

$$a^*\left((T(ab\tilde{g}))(e)\right) = \int_G abg \, d\mu_{a^*} .$$

$$\text{But } a^*\left((T(ab\tilde{g}))(e)\right) = a^*\left((T(a(b\tilde{g})))(e)\right)$$

$$= a^*\left(a(T(b\tilde{g})(e)\right)$$

$$= (a^*\cdot a)\left((T(b\tilde{g}))(e)\right)$$

$$= \int_G bg \, d\mu_{a^*\cdot a}.$$

Thus for $a, b \in A$, $a^* \in A^*$ and $g \in C_o(G)$, we have

$$\int_G abg \, d\mu_{a^*} = \int_G bg \, d\mu_{a^*\cdot a}.$$

Therefore for all $E \in \Sigma$,

$$\mu_{a^*}(E)(ab) = \mu_{a^*\cdot a}(E)(b),$$

where, by definition

$$\langle \mu_{a^*}(E) , ab \rangle = \langle \mu(E)(ab) , a^* \rangle$$

and $$\langle \mu_{a^*\cdot a}(E) , b \rangle = \langle \mu(E)(b) , a^*\cdot a \rangle$$

$$= \langle a\cdot(\mu(E)(b)), a^* \rangle$$

Therefore

$$\langle \mu(E)(ab) , a^* \rangle = \langle a\cdot(\mu(E)(b)) , a^* \rangle \text{ for all } a^* \in A^*.$$

Thus for all $E \in \Sigma$ and $a, b \in A$, we get

$$\mu(E)(ab) = a\cdot(\mu(E)(b)).$$

This proves that μ takes values in $\text{Hom}_A^{\ell}(A, A^{**})$.

Conversely, suppose $\mu : \Sigma \longrightarrow \text{Hom}_A^{\ell}(A, A^{**})$ is a vector measure of finite semi-variation which satisfies (ii), (iii) and (vi). Then for each $f \in C_o(G,A)$, we define $Tf = \mu * f$. Now by Theorem

3.2.4, T defines a continuous linear left translation invariant operator on $C_o(G,A)$ such that (v) holds. We show that T is a left $L^1(G,A)$-module homomorphism. In view of Proposition 3.2.6, it is enough to show that $T(abg) = a(T(bg))$ for all $a, b \in A$ and $g \in C_o(G)$ We have

$$(T(abg))(e) = \int ab\,\tilde{g}\,d\mu$$

$$= \lim_{\iota} \sum_{\iota} \tilde{g}(s_\iota)\,\mu(E_\iota)(ab) \quad \text{(by definition of integral)}$$

$$= \lim_{\iota} \sum_{\iota} \tilde{g}(s_\iota)\left[a\cdot(\mu(E_\iota)(b))\right]$$

$$= a\left[\lim_{\iota} \sum_{\iota} \tilde{g}(s_\iota)\,\mu(E_i)(b)\right]$$

$$= a\left[\int b\tilde{g}\,d\mu\right]$$

$$= a\left[(T(bg))(e)\right]$$

Therefore $(T(abg))(e) = a\left[(T(bg))(e)\right]$.

Also for $s \in G$,

$$(T(abg))(s) = (\tau_{s^{-1}}(T(abg)))(e)$$

$$= (T(\tau_{s^{-1}}(abg)))(e)$$

$$= (T(ab(\tau_{s^{-1}}g)))(e)$$

$$= a(Tb(\tau_{s^{-1}}g))(e)$$

$$= a(\tau_{s^{-1}}(T(bg)))(e)$$

$$= a(T(bg))(s).$$

Therefore $(T(abg))(s) = a(T(bg))(s)$.

Hence $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(C_o(G,A))$.

This completes the proof of the theorem.

Remarks 3.2.9 : (1) Let G be a unimodular locally compact group. Then as noted earlier $C_o(G,A)$ is a right $L^1(G,A)$-module. Using the arguments similar to Theorem 3.2 8, we can also characterize the space $\mathrm{Hom}^r_{L^1(G,A)}(C_o(G,A))$.

(2) The measure μ corresponding to T in Theorem 3.2 8 satisfies the condition that for each $f \in C_o(G,A)$, the mapping $s \longrightarrow \int \tau_s f \, d\mu$ vanishes at infinity. We show that if $\mu : \Sigma \longrightarrow \mathrm{Hom}_A^{\ell}(A, A^{**})$ is a vector measure satisfying

(i) For each $f \in C_o(G,A)$, $\int f \, d\mu$ exists as an element of A and (ii) There exists a finite positive regular measure ν such that

$$\left\| \int f \, d\mu \right\| \leq \int \|f\| \, d\nu,$$

then the mapping $s \longrightarrow \int \tau_s f \, d\mu$ vanishes at infinity for each $f \in C_o(G,A)$. Below, we give a proof of this result.

Since ν is regular, given $\epsilon > 0$ and $f \in C_o(G,A)$, there exists a compact subset K of G such that $\nu(G\backslash K) < \frac{\epsilon}{2\|f\|_\infty}$. Since $f \in C_o(G,A)$, there exists a compact subset K' of G such that $\|f(t)\| < \frac{\epsilon}{2\,\nu(G)}$ for $t \notin K'$. Then for $s \notin K(K')^{-1}$, we have $s^{-1}K \subseteq G\backslash K'$ and therefore

$$\left\| \int_G \tau_s f \, d\mu \right\| \leq \int_G \|\tau_s f\| \, d\nu = \int_K \|\tau_s f\| \, d\nu + \int_{G\backslash K} \|\tau_s f\| \, d\nu$$

$$\leq \int_K \|f(s^{-1}t)\| \, d\nu(t) + \|f\|_\infty \, \nu(G\backslash K)$$

$$< \frac{\epsilon}{2\,\nu(G)} \; \nu(G) + \|f\|_\infty \; \frac{\epsilon}{2\|f\|_\infty} = \epsilon.$$

Hence for each $f \in C_o(G,A)$, the mapping $s \longrightarrow \int \tau_s f \, d\mu$ vanishes at infinity

Suppose μ is a regular measure of finite variation. Then the variation $|\mu|$ of μ is finite positive regular measure such that $\left\| \int f \, d\mu \right\| \leq \int \|f\| \, d|\mu|$ for each $f \in C_o(G,A)$. Furthermore, $\int f \, d\mu$ exists as an element of A. Hence in this case, $s \longrightarrow \int \tau_s f \, d\mu$ vanishes at infinity for each $f \in C_o(G,A)$.

§3. $M(C_o(G,A))$

Let G be a locally compact Hausdorff space and A be a Banach algebra.Then $C_o(G,A)$ becomes a Banach algebra with pointwise multiplication. Lai [23] had investigated the space $M(C_o(G,A))$ under the assumption that A is a commutative Banach algebra with identity. However, there is a small gap in his proof. In this section, we characterize the space $M^l(C_o(G,A))$ when the Banach algebra A has a right approximate identity. Most of the arguments of the proof are taken from [23]. We give the proof for the sake of completeness (See also [40]).

Let $C_s^b(G,M^l(A))$ denote the space of bounded functions on G to $M^l(A)$, continuous in the strong operator topology.

Theorem 3.3.1 : Suppose G is a locally compact Hausdorff space and A is a Banach algebra with a right approximate identity Then the space $M^l(C_o(G,A))$ is isometrically isomorphic to $C_s^b(G,M^l(A))$.

Proof : Let $T \in M^l(C_o(G,A))$. Suppose $t \in G$ and $a \in A$. Let $f \in C_o(G)$ be such that $f(t) \neq 0$. We define

$$h_T(t)(a) = \frac{(Taf)(t)}{f(t)}$$

We first show that h_T is well defined

Let $\{e_\alpha\}$ be a right approximate identity of A and $g \in C_o(G)$. Then

$$\begin{aligned}(Taf)(t)\, g(t)\, e_\alpha &= ((Taf)\cdot e_\alpha g)(t)\\ &= (T(af\cdot e_\alpha g))(t)\\ &= (T(ag\cdot e_\alpha f))(t)\\ &= ((Tag)\cdot e_\alpha f)(t)\\ &= (Tag)(t)\, e_\alpha f(t)\\ &= (Tag)(t)f(t)e_\alpha.\end{aligned}$$

Since $\{e_\alpha\}$ is a right approximate identity of A, it follows that for all $f,g \in C_o(G)$ and $a \in A$,

$$(Taf)(t)g(t) = (Tag)(t)f(t) \qquad (3.1)$$

Therefore if $f(t) \neq 0$ and $g(t) \neq 0$, then

$$\frac{(Taf)(t)}{f(t)} = \frac{(Tag)(t)}{g(t)}\ .$$

This proves that h_T is well defined.

Next we show that h_T takes values in $M^{\ell}(A)$.

Suppose $t \in G$ is fixed. Choose $f \in C_o(G)$ such that $f(t) = 1$ and $\|f\|_\infty = 1$. Then for $a,b \in A$,

$$\begin{aligned}h_T(t)(ab) &= T(abf^2)(t)\\ &= T(af\cdot bf)(t)\\ &= ((Taf)\cdot bf)(t)\\ &= (Taf)(t)b\\ &= (h_T(t)(a))b.\end{aligned}$$

Thus h_T is a mapping from G to $M^{\ell}(A)$.

Also,

$$\|h_T(t)(a)\| = \|(Taf)(t)\| \leq \|Taf\|_\infty \leq \|T\| \, \|a\| \, \|f\|_\infty = \|T\| \, \|a\|.$$

Therefore

$$\|h_T(t)\| \leq \|T\| \text{ for all } t \in G.$$

Hence $$\|h_T\|_\infty \leq \|T\| \tag{3.2}$$

We shall now show that the mapping $h_T : G \longrightarrow M^{\ell}(A)$ is continuous in the strong operator topology

Let $\{t_\alpha\}$ be a net in G such that $t_\alpha \longrightarrow t_o$. Choose $f \in C_o(G)$ such that $f(t_o) = 1$ and $\|f\|_\infty = 1$. Then there exists an α_o such that $f(t_\alpha) \neq 0$ for all $\alpha \geq \alpha_o$. Now if $a \in A$ and $\alpha \geq \alpha_o$ then

$$h_T(t_\alpha)(a) = \frac{(Taf)(t_\alpha)}{f(t_\alpha)} \longrightarrow \frac{(Taf)(t_o)}{f(t_o)} = h_T(t_o)(a).$$

Therefore the map $h_T : G \longrightarrow M^{\ell}(A)$ is continuous in the strong operator topology.

Next we show that $\|T\| \leq \|h_T\|_\infty$.

It can be checked that $h_T(t)(f(t)) = (Tf)(t)$ for all $f \in C_o(G,A)$ and $t \in G$.

Thus,

$$\|(Tf)(t)\| \leq \|h_T(t)\| \, \|f(t)\|$$

$$\leq \|h_T\|_\infty \, \|f\|_\infty$$

Therefore

$$\|Tf\|_\infty \leq \|h_T\|_\infty \, \|f\|_\infty$$

and hence $$\|T\| \leq \|h_T\|_\infty \tag{3.3}$$

From (3.2) and (3.3), we get

$$\|T\| = \|h_T\|_\infty.$$

Thus corresponding to $T \in M^\ell(C_o(G,A))$, we get a mapping $h_T : G \longrightarrow M^\ell(A)$ which is continuous in the strong operator topology and $\|T\| = \|h_T\|_\infty$.

Conversely, we show that each $h \in C_s^b(G,M^\ell(A))$ gives rise to a multiplier $T \in M^\ell(C_o(G,A))$.

Let $f \in C_o(G,A)$ and $t \in G$. We define

$$(Tf)(t) = h(t)(f(t)).$$

Then Tf is a continuous function on G to A. In fact, let $\{t_\alpha\}$ be a net in G such that $t_\alpha \longrightarrow t_o$. Then

$$\|(Tf)(t_\alpha)-(Tf)(t_o)\|$$

$$= \|h(t_\alpha)(f(t_\alpha)) - h(t_o)(f(t_o))\|$$

$$\leq \|h(t_\alpha)(f(t_\alpha)) - h(t_\alpha)(f(t_o))\| + \|h(t_\alpha)(f(t_o)) - h(t_o)(f(t_o))\|$$

$$\leq \|h(t_\alpha)\| \|f(t_\alpha) - f(t_o)\| + \|h(t_\alpha)(f(t_o)) - h(t_o)(f(t_o))\|$$

$$\leq \|h\|_\infty \|f(t_\alpha) - f(t_o)\| + \|h(t_\alpha)(f(t_o)) - h(t_o)(f(t_o))\|$$

$\longrightarrow 0$ as $t_\alpha \longrightarrow t_o$, because f is continuous and h is continuous in the strong operator topology.

This proves that Tf is a continuous function.

Furthermore $\{t \in G : \|(Tf)(t)\| \geq \epsilon\} \subseteq \{t \in G : \|f(t)\| \geq \frac{\epsilon}{\|h\|_\infty}\} = E$.

Since $f \in C_o(G,A)$, E is compact. Thus it follows that $Tf \in C_o(G,A)$ for every $f \in C_o(G,A)$.

It is easy to see that $T(f.g)(t) = ((Tf).g)(t)$ for all $f, g \in C_o(G,A)$ and $t \in G$.

Therefore $T \in M^{\ell}(C_0(G,A))$.

Also it can be easily seen that $\|h\|_\infty = \|T\|$.

This completes the proof.

CHAPTER IV

MULTIPLIERS OF VECTOR VALUED L^P-SPACES

§1. INTRODUCTION

In this chapter, we study some multiplier problems of vector valued L^P-spaces.

Suppose G is a locally compact abelian group and A is a commutative Banach algebra with identity. Tewari, Dutta and Vaidya [46] proved that $M(L^1(G,A))$ is isometrically isomorphic to $M(G,A)$. Chan [4] characterized the space $M^r(L^1(G,A))$ when G is arbitrary and A is a Banach algebra with a minimal# approximate identity.

Let G be a locally compact abelian group, A be a commutative Banach algebra with identity and X be an A-module. Lai [22] characterized the space $\mathrm{Hom}_{L^1(G,A)}(L^1(G,A), L^p(G,X))$ $(1 < p < \infty)$ under the hypothesis that X^* and X^{**} have wide RNP (See, §5 of Chapter II). In Section 2, we generalize this result and give a characterization of the space $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A), L^p(G,X))$ where G is arbitrary and A is a Banach algebra with a minimal right approximate identity.

An approximate identity $\{e_\alpha\}$ is minimal if $\lim_\alpha \|e_\alpha\| = 1$.

Let G be a locally compact abelian group and Γ its dual. It is known that if T is a translation invariant operator on $L^2(G)$ then there exists a bounded measurable function ϕ on Γ such that $(Tf)\hat{} = \phi\hat{f}$ for all $f \in L^2(G)$. Conversely, every bounded measurable function on Γ defines a translation invariant operator on $L^2(G)$. The proof essentially uses the Plancherel theorem. However in the vector valued case, the Plancherel theorem does not hold in general [37]. In Section 3, we prove that if G is a compact abelian group and H is a Hilbert space then the map $f \longrightarrow \hat{f}$ is an isometry of $L^2(G,H)$ onto $\ell^2(\Gamma,H)$. Infact, for a compact group G we prove an analogue of Peter-Weyl Theorem for $L^2(G,H)$. We achieve this by using the ideas of the proof for the scalar case given in Sugiura [44]. We have included the proofs as we have not seen these results mentioned in the literature. We use these results to characterize the space of translation invariant operators on $L^2(G,H)$ where G is a compact abelian group. We also give an example to show that this characterization need not hold when H is not a Hilbert space.

§2. $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A)\ ,\ L^p(G,X))$

Let A be a Banach algebra. Arens [1] defined two products in A^{**}, the second dual of A. With respect to each of these A^{**} becomes a Banach algebra. These are described below.

For a, b $\in$ A, $a^* \in A^*$ and a^{**}, $b^{**} \in A^{**}$, define $(a^* \cdot a)$ and $(a \quad a^*) \in A^*$ by

$$(a^* \quad a)(b) = a^*(ab) \text{ and } (a \cdot a^*)(b) = a^*(ba).$$

Next, define $(a^{**} \quad a^{*})$ and $(a^{*} \quad a^{**}) \in A^{*}$ by

$$(a^{**} \cdot a^{*})(b) = a^{**}(a^{*} \cdot b) \text{ and } (a^{*} \quad a^{**})(b) = a^{**}(b \quad a^{*}).$$

Finally, define $(a^{**} \circ b^{**})$ and $(a^{**} \circ' b^{**}) \in A^{**}$ by

$$(a^{**} \circ b^{**})(a^{*}) = a^{**}(b^{**} \cdot a^{*}) \text{ and}$$

$$(a^{**} \circ' b^{**})(a^{*}) = b^{**}(a^{*} \quad a^{**})$$

It is known that $(A^{**}, \odot)$ has a right identity E of norm one if and only if A has a minimal right approximate identity. Similarly, $(A^{**}, \circ')$ has a left identity E' if and only if A has a left approximate identity [5, 47].

Throughout the section A denotes a Banach algebra with a minimal right approximate identity unless otherwise stated.

Tomiuk [47] proved that the space $M^{r}(A)$ of right multipliers of A can be embedded in A^{**}. A similar result for $M^{\ell}(A)$ is also obtained. Let π be the natural embedding of A into A^{**}. He also characterized the space $\{F \in A^{**} : a \cdot F \in \pi(A) \text{ for all } a \in A\}$. Chan [4] used the above mentioned results of Tomiuk to characterize the right multiplier space of $L^{1}(G,A)$, where G is a locally compact group.

Let G be a locally compact group and X be an A-module such that X^{**} has wide RNP. In this section, we characterize the space $\mathrm{Hom}^{\ell}_{L^{1}(G,A)}(L^{1}(G,A), L^{p}(G,X))$. To prove this result, we first modify Tomiuk's [47] results. In particular, we prove that $\mathrm{Hom}^{\ell}_{A}(A,X)$ can be embedded in X^{**}. Let π' be the natural embedding of X into X^{**}. We also obtain a characterization of the space $\{F \in X^{**} : a \cdot F \in \pi'(X) \text{ for all } a \in A\}$.

We observe that every element of $\mathrm{Hom}_A^{\ell}(\pi(A)\ ,\ \pi'(X))$ is of the form $T^{**}|_{\pi(A)}$ for some $T \in \mathrm{Hom}_A^{\ell}(A,X)$ Furthermore if $T \in \mathrm{Hom}_A^{\ell}(A,X)$ then $T^{**}|_{\pi(A)} \in \mathrm{Hom}_A^{\ell}(\pi(A)\ ,\ \pi'(X))$ and $T^{**}(\pi(a)) = \pi'(Ta)$ for $a \in A$ In the following Proposition, we show that $\mathrm{Hom}_A^{\ell}(A,X)$ can be embedded in X^{**}. For $T \in \mathrm{Hom}_A^{\ell}(A,X)$, we define an element $F^T \in X^{**}$ by

$F^T = T^{**}(E)$ where E is the right identity of A^{**}.

Proposition 4.2.1 : The mapping $T \longrightarrow F^T$ described above is an isometric isomorphism of $\mathrm{Hom}_A^{\ell}(A,X)$ into X^{**}.

Proof : It is easy to see that the mapping $T \longrightarrow F^T$ is linear

Also, $\|F^T\| = \|T^{**}(E)\| \leq \|T^{**}\|\ \|E\| = \|T\|$.

Therefore $\|F^T\| \leq \|T\|$ (2.1)

Let $a \in A$ and $x^* \in X^*$. Then

$$\langle a \cdot F^T\ ,\ x^* \rangle = \langle a \cdot T^{**}(E)\ ,\ x^* \rangle$$

$$= \langle T^{**}(a \cdot E)\ ,\ x^* \rangle$$

$$= \langle T^{**}(\pi(a) \circ E)\ ,\ x^* \rangle$$

(Note that $a \cdot E = \pi(a) \circ E$)

$$= \langle T^{**}(\pi(a))\ ,\ x^* \rangle$$

Therefore $a \cdot F^T = T^{**}(\pi(a))$

But $T^{**}(\pi(a)) = \pi'(Ta)$

Hence $a \cdot F^T = \pi'(Ta)$ (2.2)

Further $\|Ta\| = \|\pi'(Ta)\| = \|a \cdot F^T\| \leq \|a\|\ \|F^T\|$

Therefore $\|T\| \leq \|F^T\|$ (2.3)

By (2.1) and (2.3), we get that

$$\|T\| = \|F^T\|$$

Hence, $\mathrm{Hom}_A^{\ell}(A,X)$ is embedded in X^{**}.

Similarly we can embed $\mathrm{Hom}_A^{r}(A,X)$ into X^{**}.

We denote the isometric isomorphic image of $\mathrm{Hom}_A^{\ell}(A,X)$ in X^{**} under the mapping $T \longrightarrow F^T$ by $A^{\ell}_{(X)}$.

We shall need the following definitions.

Definition 4.2.2 : $N^A_{(X)} = \{F \in X^{**} : a \cdot F = 0 \text{ for all } a \in A\}$

Definition 4.2.3 : An A-module X is said to be *left order free* if for $x \in A$, $xA = 0$ implies that $x = 0$ (See [25]).

We note that if X^{**} is a left order free A-module then $N^A_{(X)} = \{0\}$. In particular, if A is a Banach algebra with a left approximate identity and $X = A$ then $N^A_{(A)} = \{0\}$. This can be seen as below (See also [47]). Let $F \in N^A_{(A)}$ and $a^* \in A^*$. Suppose $\{u_\alpha\}$ is a left approximate identity of A. Then

$$\begin{aligned}\lim_\alpha \langle u_\alpha \cdot F , a^* \rangle &= \lim_\alpha \langle \pi(u_\alpha) \odot' F , a^* \rangle \\ &= \langle E' \odot' F , a^* \rangle \\ &= \langle F , a^* \rangle .\end{aligned}$$

Since $F \in N^A_{(A)}$, it follows that $u_\alpha \cdot F = 0$ for each α. Therefore $F = 0$.

In particular, if A is a Banach algebra with identity e then $e \cdot F = F$ for $F \in X^{**}$. Therefore $N^A_{(X)} = \{0\}$.

In the following Lemma, we characterize the subspace

$\{F \in X^{**} : a \cdot F \in \pi'(X)\ \forall\ a \in A\}$ of X^{**}.

Lemma 4.2.4 : Let X be an A-module and $F \in X^{**}$. Then the following are equivalent

(i) $a \cdot F \in \pi'(X)$ for all $a \in A$.

(ii) There exist elements $T \in \mathrm{Hom}_A^{\ell}(A,X)$ and $G \in N_{(X)}^{A}$ such that

$$F = F^T + G.$$

Proof : Suppose (i) holds. For $a \in A$, we define

$T(a) = x$ where $a \cdot F = \pi'(x)$

We shall show that $T \in \mathrm{Hom}_A^{\ell}(A,X)$.

Let $a_1, a_2 \in A$. Then

$$\pi'(T(a_1a_2)) = (a_1a_2) \cdot F$$

$$= a_1 \cdot (a_2 \cdot F)$$

$$= a_1 \cdot \pi'(T(a_2)).$$

We show that $a \cdot \pi'(x) = \pi'(ax)$ for $a \in A$ and $x \in X$. Let $x^* \in X^*$. Then

$$\langle a \cdot \pi'(x) , x^* \rangle = \langle \pi'(x) , x^* \cdot a \rangle$$

$$= \langle x^* a , x \rangle$$

$$= \langle x^* , ax \rangle$$

$$= \langle \pi'(ax) , x^* \rangle$$

Therefore $a \cdot \pi'(x) = \pi'(ax)$.

Thus $\pi'(T(a_1a_2)) = \pi'(a_1T(a_2))$

Hence $T(a_1a_2) = a_1(Ta_2)$ and consequently $T \in \mathrm{Hom}_A^{\ell}(A,X)$.

Furthermore,

$$a \cdot F^T = \pi'(Ta) \qquad \text{(By (2.2))}$$

$$= a\,F \qquad \text{(By definition of T)}$$

Thus $a\,(F-F^T) = 0$ for all $a \in A$.

Therefore $G = F - F^T \in N^A_{(X)}$. Thus we have obtained an element $G \in N^A_{(X)}$ such that $F = F^T + G$.

This proves (ii).

Now suppose (ii) holds Let $F = F^T + G$ where $G \in N^A_{(X)}$. Then for $a \in A$,

$$a \cdot F = a\,F^T + a \cdot G = a\,F^T$$

But by (2.2),

$$a \cdot F^T = \pi'(Ta)$$

Therefore for all $a \in A$,

$$a \cdot F = \pi'(Ta)$$

Hence $a \cdot F \in \pi'(X)$.

This proves (i) and completes the proof of the Lemma.

We shall now use the duality result for $L^p(G,X)$ given by Lai [24] (See 2 2.1) to prove the following result.

Theorem 4.2.5 : Let G be a locally compact group and X be an A-module such that X^{**} has wide RNP with respect to the left Haar measure. Suppose $1 < p < \infty$ and $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A),\ L^p(G,X))$. Then there exists a unique $g \in L^p(G\ ,\ A^{\ell}_{(X)} + N^A_{(X)})$ such that $Tf = f * g$ for all $f \in L^1(G,A)$.

Conversely, if $g \in L^p(G , A^{\ell}_{(X)} + N^{A}_{(X)})$ then there exists a $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A) , L^p(G,X))$ such that $Tf = f * g$ for all $f \in L^1(G,A)$.

Proof : Let $\{g_\alpha\}$ be an approximate identity of $L^1(G)$ of norm 1 and $\{e_\alpha\}$ be a minimal approximate identity of A. Then $\{f_\alpha = e_\alpha g_\alpha\}$ is a minimal approximate identity of $L^1(G,A)$. Let $(\pi'(Tf_\alpha))(s) = \pi'((Tf_\alpha)(s))$ for all $s \in G$. Then $\{\pi'(Tf_\alpha)\} \subseteq L^p(G,X^{**}) = (L^{p'}(G,X^*))^*$. Thus $\{\pi'(Tf_\alpha)\}$ is a bounded net in $(L^{p'}(G,X^*))^*$. Hence by Banach-Alaglou's theorem, there exists a subnet of $\{\pi'(Tf_\alpha)\}$ (which for notational convenience we again denote by $\{\pi'(Tf_\alpha)\}$) such that $\{\pi'(Tf_\alpha)\}$ converges in the weak* topology to a function $g \in L^p(G,X^{**})$ Let $h \in L^1(G)$, $a \in A$ and $k \in L^{p'}(G,X^*)$. Then

$$T((a\ h) * f_\alpha) \longrightarrow T(a\ h).$$

Therefore $\pi'(T((a\ h)*f_\alpha) \longrightarrow \pi'(T(a\ h))$.

Next,

$$\begin{aligned}
\lim_\alpha \langle \pi'(T((a\ h) * f_\alpha)) , k\rangle &= \lim_\alpha \langle \pi'((a\ h) * (Tf_\alpha)) , k\rangle \\
&= \lim_\alpha \langle \pi'(h * a\ (Tf_\alpha)) , k\rangle \\
&= \lim_\alpha \langle h * \pi'(a\ (Tf_\alpha)) , k\rangle \\
&= \lim_\alpha \langle a \cdot \pi'(Tf_\alpha) , h*\tilde{k}\rangle \\
&= \lim_\alpha \langle \pi'(Tf_\alpha)) , (h*\tilde{k})\cdot a\rangle \\
&= \langle g , h * (k\cdot a)\tilde{} \rangle \\
&= \langle h * g , k\cdot a\rangle \\
&= \langle (a\ h) * g , k\rangle .
\end{aligned}$$

This implies that $\pi'(T((a\ h)*f_\alpha)) \xrightarrow{\text{weak}^*} (a\ h)*g$. Also $\pi'(T((a\ h)*f_\alpha)) \longrightarrow \pi'(T(a\ h))$ in the norm.

Therefore for all $a \in A$ and $h \in L^1(G)$

$$\pi'(T(a\ h)) = (a\ h)*g.$$

Since the linear span of $\{a\ h\ .\ a \in A,\ h \in L^1(G)\}$ is dense in $L^1(G,A)$, it follows that $\pi'(Tf) = f*g$ for all $f \in L^1(G,A)$. In view of the identification of X and $\pi'(X)$, we write $Tf = f*g$.

Next we show that $g \in L^p(G\ ,\ A^{\ell}_{(X)} + N^A_{(X)})$. In view of Lemma 4.2 4, it is enough to prove that $a\cdot g \in L^p(G\ ,\ \pi'(X))$ for all $a \in A$. Let $\psi : X^{**} \longrightarrow X^{**}/\pi'(X) = Y$ be the canonical mapping. Let $h_a = \psi(a\cdot g)$. Then $h_a \in L^p(G,Y) \subseteq L^p(G,Y^{**}) \subseteq (L^{p'}(G,Y^*))^*$.

Let $k \in L^{p'}(G,Y^*)$. Then it is easy to see that $g_\alpha * k \longrightarrow k$ in $L^{p'}(G,Y^*)$. Therefore by Hewitt factorization theorem, we have $L^1(G) * L^{p'}(G,Y^*) = L^{p'}(G,Y^*)$. Hence there exist functions $k_1 \in L^1(G)$ and $k_2 \in L^{p'}(G,Y^*)$ such that $k = k_1 * k_2$.

$$\text{Now } \langle k\ ,\ \tilde{h}_a \rangle = \langle k_1 * k_2\ ,\ \widetilde{(\psi(a\cdot g))} \rangle$$

$$= \langle k_2\ ,\ k_1 * \psi(a\cdot g) \rangle$$

$$= \langle k_2\ ,\ \psi((ak_1) * g) \rangle$$

$$= \langle k_2\ ,\ \psi((\pi'(T(ak_1)))) \rangle$$

$$= \langle k_2\ ,\ 0 \rangle = 0.$$

Hence $a\cdot g \in L^p(G\ ,\ \pi'(X))$ for all $a \in A$. This implies that $g \in L^p(G\ ,\ A^{\ell}_{(X)} + N^A_{(X)})$.

Conversely, suppose $g \in L^p(G\ ,\ A^{\ell}_{(X)} + N^A_{(X)})$. Then for $f \in L^1(G,A)$, define $Tf = f*g$. By Lemma 4.2.4, T defines a mapping

from $L^1(G,A)$ to $L^p(G,X)$ It is easy to see that $T \in \mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A)\ ,\ L^p(G,X))$

Furthermore, it is easy to see that $\|T\| = \|g\|_p$.

This completes the proof.

Chan [4] considered the case p = 1 and characterized the right multipliers of $L^1(G,A)$ He obtained the following result.

Theorem 4.2.6 : Let G be a locally compact group. Let $T \in M^r(L^1(G,A))$. Then there exists a unique measure $\mu \in M(G,A^{(m)})$ such that $Tf = f * \mu$. Conversely, if $\mu \in M(G,A^{(m)})$. Then T defines a right multiplier on $L^1(G,A)$ by $Tf = f * \mu$ for all $f \in L^1(G,A)$.

Here, $A^{(m)}$ denotes the isometric isomorphic image of $M^r(A)$ in A^{**} under the mapping $T \longrightarrow T^{**}(E)$. In our notation $A^{(m)} = A^{\ell}_{(A)}$ Using the ideas of the proof given by Chan [4] and applying Lemma 4.2.4, we can prove the following Theorem

Theorem 4.2.7 : Suppose G is a locally compact group and X is an A-module. Then $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A),\ L^1(G,X)) \cong M(G\ ,\ A^{\ell}_{(X)} + N^{A}_{(X)})$.

Since $L^1(G,X) \subseteq M(G,X^{**}) = (C_o(G,X^*))^*$. The duality result for $C_o(G,X)$ does not require any condition on X. Therefore in Theorem 4.2 7, we do not require X^{**} to have wide RNP.

By similar analysis, we can obtain right versions of Theorems 4.2 5 and 4.2.7.

If we take X = A, Theorems 4.2.5 and 4.2.7 reduce to the following.

Corollary 4.2.8 : Let G be a locally compact group. Then

(i) $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A)\ ,\ L^p(G,A)) \cong L^p(G\ ,\ A^{\ell}_{(A)})$ $(1 < p < \infty)$ if A^{**} has wide RNP.

(Note that $N^A_{(A)} = \{0\}$ since A has an approximate identity).

(ii) $M^r\ (L^1(G,A)) \cong M(G,\ A^{\ell}_{(A)})$. (As remarked earlier, this result was proved by Chan [4]).

If A is a Banach algebra with identity. Then we have seen that $N^A_{(X)} = \{0\}$ In this case, the space $A^{\ell}_{(X)}$ can be identified with X. This can be seen as follows.

Let $F^T \in A^{\ell}_{(X)}$. Then $e\cdot F^T = F^T$. By Lemma 4.2.4, $e\ F^T \in \pi'(X)$. Hence $F^T \in \pi'(X)$ Conversely, suppose $F \in \pi'(X)$ then $a\cdot F \in \pi'(X)$ for all $a \in A$. Again by Lemma 4 2.4, $F \in A^{\ell}_{(X)}$. Therefore $A^{\ell}_{(X)} = \pi'(X)$ Hence $A^{\ell}_{(X)}$ can be identified with X. Thus in this case, Theorems 4 2.5 and 4.2 7 reduce to the following.

Corollary 4.2.9 : Let G be a locally compact group, A be a Banach algebra with identity and X be an A-module. Then

(i) $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A)\ ,\ L^p(G,X)) \cong L^p(G,X)$, $1 < p < \infty$ if X^{**} has wide RNP.

(ii) $\mathrm{Hom}^{\ell}_{L^1(G,A)}(L^1(G,A)\ ,\ L^1(G,X)) \cong M(G,X)$

Lai [22] obtained (i) when G is a locally compact abelian group, A is a commutative Banach algebra with identity and X is an A-module such that X^* and X^{**} have wide RNP. However, one could see that his proof works if wide RNP of only X^{**} is assumed.

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§3. An Analogue of Plancherel Theorem and Translation Invariant Operators on $L^2(G,H)$

We have already observed that for a locally compact abelian group G, Plancherel theorem is the main tool for the characterization of translation invariant operators on $L^2(G)$

Let G be a compact abelian group and H be a Hilbert space. In this section, we give an analogue of Plancherel theorem for $L^2(G,H)$. We achieve this by proving an analogue of Peter-Weyl theorem for $L^2(G,H)$ on a compact group G Using the analogue of Plancherel theorem for $L^2(G,H)$, we characterize translation invariant operators on $L^2(G,H)$.

For unexplained notation and terminology, see Sugiura [44] Throughout the section, G denotes a compact group unless otherwise specified. The set of all equivalence classes of irreducible unitary representations of G is denoted by $\hat{G}$ and is called the dual object of G. We recall the Peter-Weyl Theorem, see [44].

Peter-Weyl Theorem 4.3.1 : Let $U^\lambda = (U^\lambda_{ij})$ denote a matricial unitary representation for each class $\lambda \in \hat{G}$ and let $d(\lambda)$ be the degree of U^λ. Then $\left\{ \sqrt{d(\lambda)}\ U^\lambda_{ij} : \lambda \in \hat{G},\ 1 \le i,j \le d(\lambda) \right\}$ is a complete orthonormal set in $L^2(G)$. If $H^\lambda_{ij} = \langle U^\lambda_{ij} \rangle$ is the space generated by U^λ_{ij} then $L^2(G) = \underset{\lambda \in \hat{G}}{\oplus} \overset{d(\lambda)}{\underset{i,j=1}{\oplus}} H^\lambda_{ij}$.

Definition 4.3.2 : Let H be a Hilbert space. Define an inner product in $L^2(G,H)$ by

$$\langle f,g \rangle = \int_G \langle f(s),g(s) \rangle \, ds \quad \text{for } f,g \in L^2(G,H).$$

Then $L^2(G,H)$ becomes a Hilbert space with this inner product.

We use Theorem 4 3 1 to prove the following analogue of Peter-Weyl Theorem

Theorem 4.3.3 Let H be a Hilbert space and $\{e_\alpha : \alpha \in \Lambda\}$ be a complete orthonormal set in H. Then the set

$$\left\{ \sqrt{d(\lambda)}\ e_\alpha\ U^\lambda_{ij} : \lambda \in \hat{G},\ \alpha \in \Lambda,\ 1 \le i,j \le d(\lambda) \right\}$$

is a complete orthonormal set in $L^2(G,H)$.

Let $H^{\lambda,\alpha}_{ij} = \langle e_\alpha U^\lambda_{ij} \rangle$ be the space generated by $e_\alpha\ U^\lambda_{ij}$ Then

$$L^2(G,H) = \hat{\oplus}_{\lambda\in\hat{G}} \oplus_{\alpha\in\lambda} \oplus_{i,j=1}^{d(\lambda)} H^{\lambda,\alpha}_{ij}.$$

Proof : It is easy to see that the set

$$E = \left\{ \sqrt{d(\lambda)}\ e_\alpha U^\lambda_{ij} : \lambda \in \hat{G},\ \alpha \in \Lambda,\ 1 \le i,j \le d(\lambda) \right\}$$

is orthonormal. We shall show that the set E is complete.

Since the set of finite linear sums of the form $\Sigma\ a_i f_i$, $a_i \in H$ and $f_i \in L^2(G)$ is dense in $L^2(G,H)$, it suffices to show that af belongs to the closed linear span of E for each $f \in L^2(G)$ and $a \in H$. Furthermore, since the linear span of $\{e_\alpha : \alpha \in \Lambda\}$ is dense in H, it is enough to show that $e_\alpha f$ belongs to the closed linear span of E for each $f \in L^2(G)$ and e_α, $\alpha \in \Lambda$

Let $f \in L^2(G)$ and $\epsilon > 0$. By Theorem 4.3.1, choose

$$\sum_{j=1}^{r} \sum_{k=1}^{p} \sum_{l=1}^{q} b_{jkl}\ U^{\lambda_j}_{kl}$$

where $\lambda_j \in \hat{G}$, $1 \le p, q \le \max \left\{ d(\lambda_j) : j = 1,2, \ldots r \right\}$

such that

$$\| f - \sum_{j=1}^{r} \sum_{k=1}^{p} \sum_{l=1}^{q} b_{jkl} \; U_{kl}^{\lambda_j} \|_2 < \epsilon$$

Therefore

$$\| e_\alpha f - \sum_{j=1}^{r} \sum_{k=1}^{p} \sum_{l=1}^{q} b_{jkl} \; e_\alpha \, U_{ij}^{\lambda_j} \|_2 < \epsilon.$$

This proves that $\left\{ \sqrt{d(\lambda)} \; e_\alpha \, U_{ij}^{\lambda} \cdot \lambda \in \hat{G}, \; \alpha \in \Lambda, \; 1 \le i,j \le d(\lambda) \right\}$ is a complete orthonormal set in $L^2(G,H)$ and

$$L^2(G,H) = \hat{\oplus}_{\lambda \in \hat{G}} \; \oplus_{\alpha \in \Lambda} \; \oplus_{i,j=1}^{d(\lambda)} H_{ij}^{\lambda,\alpha}$$

<u>Definition 4.3.4</u> : Let $f \in L^2(G,H)$ and $g \in L^2(G)$. Define

$$(f,g) = \int f(t) \; \overline{g(t)} \; dt$$

We note that the integral exists and defines an element of H

We also observe that $\langle f \, , \, e_\alpha \, U_{ij}^{\lambda} \rangle = \langle (f, U_{ij}^{\lambda}) \, , \, e_\alpha \rangle$.

<u>Remark 4.3.5</u> : Suppose $f \in L^2(G,H)$. By Theorem 4.3.3,

$$f = \sum_{\lambda \in \hat{G}} d(\lambda) \sum_{\alpha \in \Lambda} \sum_{i,j=1}^{d(\lambda)} \langle f \, , \, e_\alpha U_{ij}^{\lambda} \rangle \; e_\alpha U_{ij}^{\lambda}$$

$$= \sum_{\lambda \in \hat{G}} d(\lambda) \sum_{\alpha \in \Lambda} \sum_{i,j=1}^{d(\lambda)} \langle (f, U_{ij}^{\lambda}) \, , \, e_\alpha \rangle \; e_\alpha U_{ij}^{\lambda}$$

$$= \sum_{\lambda \in \hat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \left(\sum_{\alpha \in \Lambda} \langle (f, U_{ij}^{\lambda}) \, , \, e_\alpha \rangle e_\alpha \right) U_{ij}^{\lambda}$$

Therefore $f = \sum_{\lambda \in \hat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, U_{ij}^{\lambda}) \; U_{ij}^{\lambda}$

$$= \sum \sqrt{d(\lambda)} \sum_{i,j=1}^{d(\lambda)} (f, U_{ij}^{\lambda}) \; \sqrt{d(\lambda)} \; U_{ij}^{\lambda}$$

Since the set $\left\{ \sqrt{d(\lambda)}\ U^{\lambda}_{ij} : \lambda \in \hat{G},\ 1 \le i,\ j \le d(\lambda) \right\}$ is orthonormal, it follows that

$$\|f\|_2^2 = \sum_{\lambda \in \hat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \|(f, U^{\lambda}_{ij})\|^2.$$

We also remark that the definitions and techniques used in this section are adopted from the book Sugiura [44] in which scalar version of these results are discussed

Definition 4.3.6 : Let $f \in L^2(G,H)$. For $\lambda \in \hat{G}$, we define $\hat{f}(\lambda)$ as the matrix of order $d(\lambda)$ with entries from H, whose i,j^{th} entry is given by

$$\left(\hat{f}(\lambda)\right)_{ij} = (f, U^{\lambda}_{ij}).$$

The function $\hat{f}$ is called the Fourier transform of f.

Definition 4.3.7 : Let $A = [a_{ij}]$ be a $n \times n$ matrix with entries in H. We define $\|A\|^2 = \sum_{i,j=1}^{n} \|a_{ij}\|^2$.

In particular, $\|\hat{f}(\lambda)\|^2 = \sum_{i,j=1}^{d(\lambda)} \left\|\left(\hat{f}(\lambda)\right)_{ij}\right\|^2$.

Definition 4.3.8 : Let $M_n(H)$ denote the set of $n \times n$ matrices with entries in H. Let $L^2(\hat{G},H)$ be the set of all functions ϕ on the dual object $\hat{G}$ with values in the set $\bigcup_{n=1}^{\infty} M_n(H)$ such that

(i) $\phi(\lambda) \in M_{d(\lambda)}(H)$ for each $\lambda \in \hat{G}$ and

(ii) $\sum_{\lambda \in \hat{G}} d(\lambda) \|\phi(\lambda)\|^2 < \infty$.

It can be checked that $L^2(\hat{G},H)$ is a Hilbert space with the inner product defined by $\langle \phi , \psi \rangle = \sum_{\lambda \in \hat{G}} d(\lambda)\ \text{Trace}\left(\phi(\lambda)\cdot(\psi(\lambda))^T\right)$ where $\left(\psi(\lambda)\right)^T$ denotes the transpose of $\psi(\lambda)$ Also if $A = [a_{ij}]$, $B = [b_{ij}] \in M_n(H)$ then $\left(A\cdot B^T\right)_{ij} = \sum_k \langle a_{ik} , b_{jk} \rangle$. Thus $A\,B^T$ is a $n \times n$ scalar matrix.

Note that if $\phi \in L^2(\hat{G},H)$ then $\|\phi\|^2 = \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d(\lambda)} d(\lambda) \left\|\left(\phi(\lambda)\right)_{ij}\right\|^2$

$$= \sum_{\lambda \in \hat{G}} d(\lambda)\ \|\phi(\lambda)\|^2$$

We now prove the following Theorem.

Theorem 4.3.9: The Fourier transform $f \longrightarrow \hat{f}$ is an isometry of the Hilbert space $L^2(G,H)$ onto $L^2(\hat{G},H)$.

Proof: Since $\|f\|_2^2 = \sum_{\lambda \in \hat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \|(f,U_{ij}^{\lambda})\|^2$ (See Remark 4.3.5). Therefore the mapping $f \longrightarrow \hat{f}$ is an isometry of $L^2(G,H)$ into $L^2(\hat{G},H)$. We shall show that this mapping is onto.

Suppose $\phi \in L^2(\hat{G},H)$. Then the series

$$\sum_{\lambda \in \hat{G}} d(\lambda) \sum_{i,j=1}^{d(\lambda)} \left(\phi(\lambda)\right)_{ij}\ U_{ij}^{\lambda}(t)$$

$$= \sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d(\lambda)} \sqrt{d(\lambda)}\ \left(\phi(\lambda)\right)_{ij}\ \left(\sqrt{d(\lambda)}\ U_{ij}^{\lambda}(t)\right)$$

defines an element $f \in L^2(G,H)$, since

$$\sum_{\lambda \in \hat{G}} \sum_{i,j=1}^{d(\lambda)} d(\lambda) \left\|\left(\phi(\lambda)\right)_{ij}\right\|^2 = \sum_{\lambda \in \hat{G}} d(\lambda)\|\phi(\lambda)\|^2 < \infty.$$

Also $(f, U^{\lambda}_{ij}) = \left(\phi(\lambda)\right)_{ij}$ for $1 \leq i, j \leq d(\lambda)$.

Thus we have obtained an element $f \in L^2(G,H)$ for which $\hat{f}(\lambda) = \phi(\lambda)$

If we take G to be a compact abelian group then we get an analogue of Plancherel Theorem for $L^2(G,H)$ which states:

Theorem 4.3.10: Let G be a compact abelian group with dual Γ and H be a Hilbert space Then the Fourier transform $f \longrightarrow \hat{f}$ is an isometry of $L^2(G,H)$ onto $\ell^2(\Gamma,H)$.

The following example shows that the Theorem 4.3.10 need not hold if H is not a Hilbert space.

Example [37]: Let $\mathbb{T}$ be the circle group and $p > 2$. Choose a sequence $\{ \lambda_n \} \in \ell^p \setminus \ell^2$.

Define $F : \mathbb{T} \longrightarrow \ell^p$ by

$F(t) = \{\lambda_n e^{int}\}$ Then $F \in L^2(\mathbb{T}, \ell^p)$ and

$$\hat{F}(k) = \int F(t)\, e^{-ikt}\, dt = \int \{ \lambda_n e^{int} \} e^{-ikt}\, dt$$

$$= \left\{\lambda_n \delta_{kn}\right\}_{n=-\infty}^{\infty}$$

$$\|\hat{F}(k)\|_{\ell^p} = |\lambda_k|.$$

Therefore $\sum_k \|\hat{F}(k)\|^2_{\ell^p} = \sum_k |\lambda_k|^2 = \infty$.

Hence $\hat{F} \notin \ell^2(\ell^p)$.

Remark 4.3.11 : F.J. Ruiz Blasco and J.L. Torrea [37] had obtained the Plancherel theorem for the spaces $V^2_{c,B}(G)$ (introduced by Phillips [32]) which contain the spaces $L^2(G,B)$ where G is a

locally compact abelian group and B is a Banach space.

We use Theorem 4.3.10 to give a characterization of translation invariant operators on $L^2(G,H)$

Theorem 4.3.12: Suppose G is a compact abelian group with dual Γ and H is a Hilbert space. Let T be a bounded linear translation invariant operator on $L^2(G,H)$. Then there exists a bounded function $\phi : \Gamma \longrightarrow \mathcal{L}(H)$ such that $(Tf)\hat{}(\gamma) = \phi(\gamma)(\hat{f}(\gamma))$ for all $f \in L^2(G,H)$

Conversely, every bounded function from Γ to $\mathcal{L}(H)$ defines a bounded linear translation invariant operator on $L^2(G,H)$.

Proof : Suppose T is a translation invariant operator on $L^2(G,H)$, $f \in L^1(G)$ and $g \in L^2(G,H)$. Then

$$\begin{aligned} T(f*g) &= T\left(\int f(y)\, \tau_y g \, d\lambda(y)\right) \\ &= \int f(y)\, T(\tau_y g)\, d\lambda(y) \\ &= \int f(y)\, \tau_y(Tg)\, d\lambda(y) \\ &= f*Tg. \end{aligned}$$

Thus we see that,

$$T(f*g) = f*Tg \text{ for } f \in L^1(G) \text{ and } g \in L^2(G,H) \qquad (3.1)$$

Let $\gamma \in \Gamma$. Choose $f \in L^2(G)$ such that $\hat{f}(\gamma) = 1$. For $x \in H$, we define

$$\phi(\gamma)(x) = (T(xf))\hat{}\,(\gamma).$$

In view of (3.1), it follows that ϕ is well defined. Further,

$$\|\phi(\gamma)(x)\| = \|T(x\gamma)\hat{}\,(\gamma)\| \leq \|T(x\gamma)\|_1 \leq \|T(x\gamma)\|_2 \leq \|T\|\,\|x\|.$$

Therefore $\phi : \Gamma \longrightarrow \mathcal{L}(H)$ is a bounded function. Let $g \in L^2(G)$.

Then

$$(T(xg))\hat{}\ (\gamma) = \hat{f}(\gamma)\ (T(xg))\hat{}\ (\gamma)$$

$$= (f*T(xg))\hat{}\ (\gamma)$$

$$= (T(f*(xg)))\hat{}\ (\gamma)$$

$$= (g*T(xf))\hat{}\ (\gamma)$$

$$= \hat{g}(\gamma)\ \phi(\gamma)(x)$$

$$= \phi(\gamma)\ (x\hat{g}(\gamma)).$$

Since the linear span of $\{xg : x \in H,\ g \in L^2(G)\}$ is dense in $L^2(G,H)$, we have

$$(Tf)\hat{}(\gamma) = \phi(\gamma)(\hat{f}(\gamma)) \text{ for all } f \in L^2(G,H).$$

Conversely, suppose ϕ . $\Gamma \longrightarrow \mathfrak{L}(H)$ is a bounded function For $f \in L^2(G,H)$, define

$$\psi(\gamma) = \phi(\gamma)(\hat{f}(\gamma)).$$

Then it is easy to see that $\psi \in \ell^2(\Gamma,H)$. By Theorem 4.3 10, there exists a unique function $F \in L^2(G,H)$ such that $\hat{F}(\gamma) = \psi(\gamma)$ for all $\gamma \in \Gamma$. Define $Tf = F$.

Thus we get a mapping $T : L^2(G,H) \longrightarrow L^2(G,H)$ such that

$$(Tf)\hat{}(\gamma) = \phi(\gamma)\ (\hat{f}(\gamma)) \text{ for all } f \in L^2(G,H) \text{ and } \gamma \in \Gamma$$

It is easy to see that T is linear and commutes with translations. We shall show that T is continuous.

By Theorem 4.3.10, we have

$$\|Tf\|_2 = \|(Tf)\hat{}\|_2 \text{ for all } f \in L^2(G,H).$$

But $$\|(Tf)\hat{}\|_2 = (\sum_\gamma \|\phi(\gamma)(\hat{f}(\gamma))\|^2)^{1/2}$$

$$\leq (\sum_\gamma \|\phi(\gamma)\|^2 \|\hat{f}(\gamma)\|^2)^{1/2}$$

$$\leq \|\phi\|_\infty \|\hat{f}\|_2$$

$$= \|\phi\|_\infty \|f\|_2$$

Therefore $\|Tf\|_2 \leq \|\phi\|_\infty \|f\|_2$

Thus the bounded function ϕ from Γ to $\mathfrak{L}(H)$ defines a bounded linear translation invariant operator T on $L^2(G,H)$ such that $(Tf)\hat{}(\gamma) = \phi(\gamma)(\hat{f}(\gamma))$ for all $f \in L^2(G,H)$ and $\gamma \in \Gamma$.

The following example shows that Theorem 4 3.12 need not be true if $\mathcal{H}$ is not a Hilbert space.

Example: Let $G = \mathbb{T}$ be the circle group and $B = \ell^p$, $p < 2$.
We shall define a bounded mapping ϕ: $\Gamma \longrightarrow \mathfrak{L}(B)$ and construct a function $f \in L^2(G,B)$ for which there does not exist any $f \in L^2(G,B)$ such that $\hat{F}(\gamma) = \phi(\gamma)\ (\hat{f}(\gamma))$ for all $\gamma \in \Gamma$

Let x be an arbitrary element of ℓ^p. For $k \in \mathbb{Z}$, define $\phi(k)(x)$ to be the element of ℓ^p in which o^{th} and k^{th} entries of x are interchanged.

More precisely,

$$[\phi(k)(x)]_m = x_m \text{ if } m \neq 0 \text{ or } k$$

$$= x_o \text{ if } m = k$$

$$= x_k \text{ if } m = 0.$$

Then $\|\phi(k)(x)\|_p = \|x\|_p$. Therefore $\phi(k) \in \mathfrak{L}(B)$ for each $k \in \mathbb{Z}$ and $\|\phi\|_\infty = 1$. Thus the mapping ϕ: $\Gamma \longrightarrow \mathfrak{L}(B)$ is bounded.

Let e_o be the element of ℓ^p such that

$$(e_o)_m = 1 \text{ if } m = 0$$

$$= 0 \text{ otherwise.}$$

Choose a sequence $\{a_k\} \in \ell^2 \setminus \ell^p$. Let $g \in L^2(G)$ be such that $\hat{g}(k) = a_k$ .Then

$$\{\phi(k)\ (e_o \hat{g}(k))\}_m = \hat{g}(k) \text{ if } m = k$$

$$= 0 \text{ otherwise}$$

Let $H(t) = \{a_n\ e^{int}\}_{-\infty}^{\infty}$. then $H(t) \notin \ell^p$ for all $t \in G$ Hence $H \notin L^2(G, \ell^p)$ but $H \in L^2(G, \ell^2)$. Further,

$$\hat{H}(k) = \int \{a_n\ e^{int}\}\ e^{-ikt}\ dt$$

$$= \{a_n\ \delta_{kn}\}_{-\infty}^{\infty}$$

Therefore $\hat{H}(k) = \phi(k)\ (e_o \hat{g}(k))$, for all $k \in \mathbb{Z}$

Let $f = e_o g$ Then by uniqueness of the Fourier transform, there does not exist any $F \in L^2(G, \ell^p)$ such that $\hat{F}(k) = \phi(k)(\hat{f}(k))$.

CHAPTER V

ISOMETRIC ISOMORPHISMS

§1. Introduction

Several isomorphism theorems have been obtained for various algebras of functions and measures on locally compact groups. Let G, G_1, G_2 be locally compact groups Wendel [51] proved that if there exists a norm decreasing isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ then the groups G_1 and G_2 are topologically isomorphic. Strichartz [43] and independently Parrott [29] considered the situation when the groups G_i's are compact and established the result when $L^p(G_1)$ and $L^p(G_2)$, $1 \leq p < \infty$, $p \neq 2$ are isometrically isomorphic. Gaudry [14] proved the same assertion from the hypothesis that there exists an isometric isomorphism of $M(L^p(G_1))$ onto $M(L^p(G_2))$. The crucial step in solving these problems was to give a characterization of isometric multipliers of $L^p(G)$. It is known that for $1 \leq p < \infty$, $p \neq 2$, the isometric right multipliers of $L^p(G)$ are of the form $c\,\delta_t$ where c is a scalar such that $|c| = 1$ and δ_t is the Dirac measure with unit mass at $t \in G$ (See [29] and [51]). Parthasarathy and Tewari [31] proved the result for a class of Segal algebras.

The above results motivated us to develop a similar theory in the vector valued case. In Section 2, we determine surjective isometric multipliers of $L^1(G,A)$. Let G be a locally compact

abelian group and A be a commutative semi-simple Banach algebra with a minimal approximate identity. We prove that surjective isometric multipliers of $L^1(G,A)$ are of the form $F\,\delta_t$ where F is an isometric multiplier of A onto itself and $t \in G$.

We know that if the algebra A has identity then multipliers of A are given by elements of A. Let $I(A) = \{a \in A: a$ defines an isometric multiplier on $A\}$. Precisely, $I(A) = \{a \in A: a$ is invertible, $\|a\| = \|a^{-1}\| = 1\}$. Thus in the case when A has an identity, surjective isometric multipliers of $L^1(G,A)$ are of the form $a\,\delta_t$ where $a \in I(A)$ and $t \in G$.

Throughout the Chapter, by multipliers of $L^p(G,A)$. we shall mean $L^1(G,A)$-module homomorphisms of $L^p(G,A)$.

Let G be a measure space and X be a Banach space. We say that $L^p(G,X)$ satisfies the property 'C' if every isometry of $L^p(G,X)$ maps functions of almost disjoint supports to functions of almost disjoint supports. Let G be a locally compact abelian group and A be a commutative Banach algebra with a minimal approximate identity. In Section 3, we determine surjective isometric multipliers of $L^p(G,A)$, under the assumption that $1 < p < \infty$, $p \neq 2$ and $L^p(G,A)$ satisfies the property 'C'. In fact, we prove that every surjective isometric multiplier of $L^p(G,A)$ is of the form $F\,\delta_t$ where F is an isometric multiplier of A onto itself and $t \in G$. We also give the conditions on a measure space G and a Banach space X such that $L^p(G,X)$ satisfies the property 'C'.

Let A_1, A_2 be commutative semi-simple Banach algebras with identity such that A_1 does not contain ℓ_1^2. In other words, A_1 does

not contain any two vectors x, y such that $\|\alpha x + \beta y\| = |\alpha| \|x\| + |\beta| \|y\|$ for all scalars α and β. Let $I'(A_1)$ denote the closure of the linear span of $I(A_1)$. Note that $I'(A_1)$ is a Banach algebra. In Section 4, we prove that if there exists an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ then the groups G_1, G_2 are topologically isomorphic and the algebras $I'(A_1)$, $I'(A_2)$ are isometrically isomorphic.

Wendel [51] gave an explicit characterization of isometric isomorphisms of $L^1(G_1)$ onto $L^1(G_2)$. Let the algebras A_1 be such that $I'(A_1)$ is equal to A_1. We prove that if ψ is an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ then there exists a topological isomorphism ϕ of G_1 onto G_2, an isometric isomorphism F of A_1 onto A_2 and a continuous mapping β of G_1 into $I(A_2)$ such that $(\psi f)(\phi(s)) = c\,\beta(s)F(f(s))$ a.e. for all $f \in L^1(G_1,A_1)$ where c is a unique positive constant. Conversely if ϕ, F, β and c are as above then ψ defined by $(\psi f)(\phi(s)) = c\,\beta(s)F(f(s))$ a.e. for $f \in L^1(G_1,A_1)$, is an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$.

Strichartz [42] and independently Johnson [18] proved that every isometric isomorphism of $M(G_1)$ onto $M(G_2)$ maps $L^1(G_1)$ onto $L^1(G_2)$. We prove, using different techniques, that every isometric isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$ maps $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$. Conversely, every isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ can be extended to $M(G_1,A_1)$ as an isometric isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$.

§2. Isometric Multipliers of $L^1(G,A)$

Let G be a locally compact abelian group and A be a commutative semisimple Banach algebra with a minimal approximate identity. In this section, we give a characterization of isometric multipliers of $L^1(G,A)$. We shall need the following characterization of multipliers of $L^1(G,A)$ due to Chan [4]. We state the result in the form in which we shall need it.

Theorem 5.2.1 : Let T be a multiplier on $L^1(G,A)$. Then there exists a unique $\mu \in M(G,M(A))$ such that $Tf = f * \mu$ for all $f \in L^1(G,A)$

Conversely, every measure $\mu \in M(G,M(A))$ defines a multiplier on $L^1(G,A)$ by $Tf = f * \mu$ for all $f \in L^1(G,A)$

Furthermore, $\|T\| = \|\mu\|$.

We shall also need the following definitions and observations.

Definition 5.2.2 : Let X be a Banach space A measure $\mu \in M(G,X)$ is said to be *discrete* if it is concentrated on a countable set. The set of all discrete measures in $M(G,X)$ is denoted by $M_d(G,X)$.

A measure $\mu \in M(G,X)$ is said to be *continuous* if $\mu(\{s\}) = 0$ for every $s \in G$. The set of all continuous measures in $M(G,X)$ is denoted by $M_c(G,X)$.

Observations 5.2.3 : (1) Each $\mu \in M_d(G,X)$ is of the form $\mu = \sum_n x_n \delta_{t_n}$ where t_n's are distinct. We also have $\|\mu\| = \sum_n \|x_n\|$.

(2) It can be seen as in the scalar case that each $\mu \in M(G,X)$ can

be written as $\mu = \mu_c + \mu_d$ where $\mu_c \in M_c(G,X)$, $\mu_d \in M_d(G,X)$ and $\|\mu\| = \|\mu_c\| + \|\mu_d\|$.

(3) Suppose G is a locally compact abelian group and A is a commutative semi-simple Banach algebra Since A is semi-simple, it follows that $L^1(G,A)$ is semi-simple [19]. Let T be an isometric multiplier of $L^1(G,A)$ onto itself. Suppose Γ denote the dual of G and $\Delta(A)$ denote the maximal ideal space of A Let $f \in L^1(G,A)$, $\gamma \in \Gamma$ and $\phi \in \Delta(A)$. Then

$$|(\mathcal{F}Tf)(\gamma,\phi)| = |(\mathcal{F}f)(\gamma,\phi)| \qquad [31]$$

We shall need the above observations to prove the following result.

Theorem 5.2.4 : Let G be a locally compact abelian group and A be a commutative semi-simple Banach algebra with a minimal approximate identity. Suppose T is an isometric multiplier of $L^1(G,A)$ onto itself. Then the measure corresponding to T given as in Theorem 5.2.1 is of the form $F\,\delta_t$ where F is an isometric multiplier of A onto itself and $t \in G$.

Conversely, if μ is of the form $F\,\delta_t$ where F is an isometric multiplier of A onto itself and $t \in G$ then μ defines an isometric multiplier of $L^1(G,A)$ onto itself.

Proof: Let T^{-1} denote the inverse of T. Then T^{-1} defines an isometric multiplier of $L^1(G,A)$ onto itself. Let μ and μ' be the measures corresponding to T and T^{-1} respectively. Then for $f \in L^1(G,A)$,

Therefore,

$$1 = \left|\frac{\phi\ (\hat{\mu}(\gamma)(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right| = \left|\frac{\phi\ (\sum_n \overline{\gamma(t_n)}\ F_n(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right|$$

$$\leq \left|\frac{\overline{\gamma(t_1)}\ \phi(F_1(\hat{f}(\gamma))) + \overline{\gamma(t_2)}\ \phi(F_2(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right| + \sum_{n>2} \left|\frac{\overline{\gamma(t_n)}\ \phi(F_n(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right|$$

$$\leq \sum_n \left|\frac{\overline{\gamma(t_n)}\ \phi(F_n(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right|$$

$$= \sum_n \left|\frac{\phi\ (F_n(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right|$$

$$\leq \sum_n \|F_n\| = 1 \qquad \text{(cf Larsen [27, p. 19])}.$$

Therefore

$$\left|\frac{\overline{\gamma(t_1)}\ \phi(F_1(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))} + \frac{\overline{\gamma(t_2)}\ \phi(F_2(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right| = \left|\frac{\overline{\gamma(t_1)}\ \phi(F_1(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right|$$

$$+ \left|\frac{\overline{\gamma(t_2)}\ \phi(F_2(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))}\right|$$

This implies that

$$\frac{\overline{\gamma(t_1)}\ \phi(F_1(\hat{f}(\gamma)))}{\phi(\hat{f}(\gamma))} \left(\overline{\frac{\overline{\gamma(t_2)}\ \phi(F_2(\hat{f}(\gamma)))}{\phi(\hat{f}(y))}}\right) \geq 0$$

Thus for all $\gamma \in \Gamma$ and $\phi \in \Delta(A)$, we have

$$\gamma(t_2-t_1)\ \phi\ (F_1(\hat{f}(\gamma)))\ \overline{\phi\ (F_2(\hat{f}(\gamma)))} \geq 0$$

Let $a \in A$ be such that $\phi(a) \neq 0$ Then

$$\gamma(t_2-t_1)\ \phi\ (a\ F_1(\hat{f}(\gamma)))\ \overline{\phi\ (a\ F_2(\hat{f}(\gamma)))} \geq 0$$

Since F_1, $F_2 \in M(A)$ and $\phi(\hat{f}(\gamma)) \neq 0$, it follows that

$$\gamma(t_2-t_1)\ \phi\ (F_1(a))\ \overline{\phi\ (F_2(a))} \geq 0 \text{ for all } \gamma \in \Gamma.$$

In particular, if we take γ to be the identity of Γ then

$$\phi\ (F_1(a))\ \overline{\phi\ (F_2(a))} \geq 0.$$

Therefore, we get

$$\gamma(t_2-t_1) \geq 0 \text{ for all } \gamma \in \Gamma.$$

It follows that $t_1 = t_2$.

This proves that $\mu = F\ \delta_t$ where $F \in M(A)$ and $t \in G$. Since $\|\mu\| = 1$, it follows that $\|F\| = 1$. Also μ is invertible and $\|\mu^{-1}\| = 1$. Thus it follows that F is invertible and $\mu^{-1} = F^{-1}\ \delta_{-t}$ and hence $\|F^{-1}\| = 1$ This proves that F is an isometric multiplier of A onto itself.

Conversely, suppose μ is of the form $F\ \delta_t$ where F is an isometric multiplier of A onto itself and $t \in G$. We define $Tf = f * \mu$ for all $f \in L^1(G,A)$. It is easy to see that T is an isometric multiplier of $L^1(G,A)$ onto itself.

Remarks 5.2.5 : (1) In Theorem 5.2.4, we characterized the surjective isometric multipliers of $L^1(G,A)$. It is known that if A is a commutative, semi-simple, regular and tauberian Banach algebra then so is $L^1(G,A)$. Thus by Theorem 1 of [31], every isometric multiplier of $L^1(G,A)$ is surjective.

(2) The arguments for the proof of Theorem 5 2 4 go through if instead of $L^1(G,A)$ we take Segal algebra $S(G,A)$ consisting of vector valued functions whose multiplier space is isomorphic to $M(G,M(A))$. In particular, if A has identity then surjective isometric multipliers of $S(G,A)$ are of the form $a\,\delta_t$ where $a \in I(A)$ and $t \in G$

We define Segal algebras consisting of vector functions and quote some properties of $S(G,A)$ needed for the above result.

Definition 5.2.6 : Let $S(G,A)$ be a subalgebra of $L^1(G,A)$ satisfying the following properties.

(i) $S(G,A)$ is dense in $L^1(G,A)$ in the norm topology of $L^1(G,A)$ and is a Banach algebra with some norm $\|\cdot\|_s$.

(ii) $S(G,A)$ is invariant under translations and $\|\tau_t f\|_s = \|f\|_s$ for every $t \in G$ and $f \in S(G,A)$.

(iii) For every $f \in S(G,A)$, the mapping $t \longrightarrow \tau_t f$ is continuous from G into $S(G,A)$ with $\|\ \|_s$ norm.

(iv) For $f \in S(G,A)$ and $a \in A$, $af \in S(G,A)$ and $\|af\|_s \leq \|a\|\ \|f\|_s$.

Some properties of S (G,A) 5.2.7 [50] :

(1) There exists a constant c such that $\|f\|_1 \leq c\,\|f\|_s$ for all $f \in S(G,A)$

(2) Let $f \in S(G,A)$ and $\mu \in M(G,A)$ then $f * \mu \in S(G,A)$ and $\|f * \mu\|_s \leq \|f\|_s\ \|\mu\|$.

In particular, $S(G,A)$ is an ideal of $L^1(G,A)$.

(3) The maximal ideal space of $S(G,A)$ is homeomorphic to $\Gamma \times \Delta(A)$.

(4) If A has identity then the set $\{f \in L^1(G,A) : \mathcal{F}f$ has compact support$\}$ is dense in $S(G,A)$

§3. Isometric Multipliers of $L^p(G,A)$

Let G be a locally compact abelian group and A be a commutative semi-simple Banach algebra with a minimal approximate identity. In this section, we characterize surjective isometric multipliers of $L^p(G,A)$ under the assumption that $1 < p < \infty$, $p \neq 2$ and $L^p(G,A)$ satisfies the property 'C'.

Let G be a measure space and X be a Banach space We discuss below some conditions on G and X such that $L^p(G,X)$ satisfies the property 'C'.

Suppose G is a σ-finite measure space and X is a separable Banach space which is not an ℓ^p-direct sum of two non-zero Banach spaces. Then it follows from the proof of Theorem 5.2 of [41] that $L^p(G,X)$ satisfies the property 'C'.

If H is a Hilbert space then we prove that $L^p(G,H)$ satisfies the property 'C'. The proof uses the arguments similar to those given for the proof of Lemma 22 [35, P. 415], See also [3].

Proposition 5.3.1 : Let G be a measure space and H be a Hilbert space. Suppose $1 \leq p < \infty$, $p \neq 2$ and T is an isometry on $L^p(G,H)$ If $f, g \in L^p(G,H)$ have almost disjoint supports then so do Tf and Tg.

Proof : We first show that f and g have almost disjoint supports if and only if

$$\|f+g\|_p^p + \|f-g\|_p^p = 2\,\|f\|_p^p + 2\,\|g\|_p^p \qquad (3\ 1)$$

It is easy to see that if f and g have almost disjoint supports then (3.1) holds.

Conversely, suppose (3.1) holds Then

$$\int\left[\|f(s) + g(s)\|^p + \|f(s)-g(s)\|^p\right]ds = 2\int\left[\|f(s)\|^p + \|g(s)\|^p\right]ds \qquad (3.2)$$

Note that if $a,b \geq 0$ then

$$a^p + b^p \leq (a+b)^p \text{ for } p \geq 1 \qquad (3.3)$$

and

$$a^p + b^p \geq (a+b)^p \text{ for } 0 \leq p < 1 \qquad (3.4)$$

Suppose $\frac{p}{2} > 1$ and $s \in G$ such that $f(s) \neq 0$ and $g(s) \neq 0$ Since the map $t \longrightarrow t^r$ for $r > 1$ is strictly convex, we have

$$\frac{1}{2}\left(\|f(s) + g(s)\|^2\right)^{\frac{p}{2}} + \frac{1}{2}\left(\|f(s) - g(s)\|^2\right)^{\frac{p}{2}}$$

$$> \left[\frac{1}{2}\|f(s) + g(s)\|^2 + \frac{1}{2}\|f(s) - g(s)\|^2\right]^{\frac{p}{2}}$$

$$= \left[\|f(s)\|^2 + \|g(s)\|^2\right]^{\frac{p}{2}}$$

$$\geq \|f(s)\|^p + \|g(s)\|^p \qquad \text{(By (3.3))}$$

Therefore for $p > 2$, we have

$$\|f(s) + g(s)\|^p + \|f(s) - g(s)\|^p > 2\,\|f(s)\|^p + 2\,\|g(s)\|^p$$

whenever $f(s) \neq 0$ and $g(s) \neq 0$. Thus if (3.2) holds, for almost all $s \in G$ either $f(s) = 0$ or $g(s) = 0$ Hence f and g have almost disjoint supports.

Now suppose $\frac{p}{2} < 1$ and $s \in G$ such that $f(s) \neq 0$ and $g(s) \neq 0$ Using the strict concavity of the map $t \longrightarrow t^r$ for $r < 1$ and (3 4), we again conclude that if (3.2) holds then f and g have almost disjoint supports.

Now the proof of the Proposition is an easy consequence of (3 1). If f and g have almost disjoint supports then (3.1) holds. Since T is an isometry, it follows that

$$\|Tf + Tg\|_p^p + \|Tf - Tg\|_p^p = 2\,\|Tf\|_p^p + 2\,\|Tg\|_p^p$$

It again follows from (3.1) that Tf and Tg have almost disjoint supports.

This completes the proof of the Proposition.

Throughout rest of the section, G denotes a locally compact abelian group and A denotes a commutative Banach algebra with a minimal approximate identity. Suppose $L^p(G,A)$ satisfies the property 'C '. We characterize surjective isometric multipliers of $L^p(G,A)$ In fact, we prove that every surjective isometric multiplier on $L^p(G,A)$ maps $L^1 \cap L^p(G,A)$ onto $L^1 \cap L^p(G,A)$ and preserves the L^1-norm. Consequently it can be extended uniquely to $L^1(G,A)$ as a surjective isometric multiplier of $L^1(G,A)$. Using the characterization of surjective isometric multipliers of $L^1(G,A)$ obtained in section 2, we get the required characterization of surjective isometric multipliers of $L^p(G,A)$. To prove this result, we first prove a Lemma. We shall need the following result for the Lemma whose proof is similar to the proof of Proposition 3.2.6 and is therefore omitted.

Proposition 5.3.2 : Let T be a continuous linear operator on $L^p(G,A)$ $(1 \leq p < \infty)$ Then the following are equivalent.

(i) $T \in \mathrm{Hom}_{L^1(G,A)}(L^p(G,A))$

(ii) $T\tau_s = \tau_s T$ and $T(af) = a(Tf)$ for all $f \in L^p(G,A)$, $a \in A$ and $s \in G$

We now prove the following Lemma. By λ we denote the Haar measure on G

Lemma 5.3.3: Let T be an isometric multiplier of $L^p(G,A)$. Let $a \in A$ and C be a measurable subset of G such that $\lambda(C) < \infty$ Then for almost all $s \in G$ there exists a measurable subset D of G with $\lambda(C) = \lambda(D)$ and elements $b(s) \in A$ with $\|b(s)\| = \|a\|$ such that

$$(T(a\,\chi_C))(s) = b(s)\,\chi_D(s).$$

Proof : We take a representative of the class $T(a\,\chi_C) \in L^p(G,A)$ and denote it by $T(a\,\chi_C)$ itself. We define

$$E(s) = \frac{(T(a\,\chi_C))(s)}{\|T(a\,\chi_C)(s)\|} \text{ if } (T(a\,\chi_C))(s) \neq 0$$

$$= x \text{ otherwise, where } x \in A \text{ with } \|x\| = 1$$

Therefore

$$(T(a\,\chi_C))(s) = E(s)\,\|T(a\,\chi_C))(s)\| \qquad (3.6)$$

and $\|E(s)\| = 1$ for all $s \in G$.

Let $\sum_{i=1}^{n} \alpha_i \chi_{C_i}$ be a scalar valued simple function where C_i's are pairwise disjoint We define

$$(F(\sum_{i=1}^{n} \alpha_i \chi_{C_i}))(s) = \frac{1}{\|a\|} \sum_{i=1}^{n} \alpha_i \|(Ta\,\chi_{C_i}))(s)\| \text{ for almost all } s \in G.$$

Since $L^p(G,A)$ satisfies the property 'C', $T(a\,\chi_{C_i})$'s $(1 \le i \le n)$ have almost pairwise disjoint supports and therefore

$$\|F(\sum_{i=1}^{n} \alpha_i \chi_{C_i})\|_p^p = \frac{1}{\|a\|^p} \sum_{i=1}^{n} |\alpha_i|^p \|T(a\,\chi_{C_i})\|_p^p$$

$$= \frac{1}{\|a\|^p} \sum_{i=1}^{n} |\alpha_i|^p \|a\,\chi_{C_i}\|_p^p \quad \text{(since T is an isometry)}$$

$$= \|\sum_{i=1}^{n} \alpha_i \chi_{C_i}\|_p^p.$$

Hence $\|F(\sum_{i=1}^{n} \alpha_i \chi_{C_i})\|_p = \|\sum_{i=1}^{n} \alpha_i \chi_{C_i}\|_p$.

Note that F is linear on simple functions. By Proposition 5.3 2, T commutes with translations Therefore F commutes with translations. Hence F can be extended to $L^p(G)$ as an isometric translation invariant operator. By Theorem 1 of [31], there exists a constant $\alpha \in \mathbb{C}$ of modulus one and $s_o \in G$ such that $F = \alpha\,\tau_{s_o}$.

By definition,

$$(F(\chi_C))(s) = \frac{1}{\|a\|} \|(T(a\,\chi_C))(s)\| \text{ for almost all } s \in G.$$

Also,

$$(F(\chi_C))(s) = \tau_{s_o}\chi_C(s) \quad \text{(Note that } \alpha = 1 \text{ since } \alpha > 0 \text{ and } |\alpha|=1)$$

$$= \chi_C(s_o^{-1}s)$$

$$= \chi_{s_oC}(s).$$

Therefore,

$$\|(T(a\,\chi_C))(s)\| = \|a\|\,\chi_{s_oC}(s) \text{ a.e.}$$

But from (3.6)

$$(T(a\,\chi_C))(s) = E(s)\,\|T(a\,\chi_C))(s)\|$$

$$= E(s)\,\|a\|\,\chi_{s_oC}(s) \text{ a.e.}$$

Let $b(s) = E(s)\,\|a\|$ and $D = s_oC$.

Thus $(T(a\,\chi_C))(s) = b(s)\,\chi_D(s)$ a e., where

$\|b(s)\| = \|a\|$ for all $s \in G$ and $\lambda(C) = \lambda(D)$.

This completes the proof of the Lemma.

Theorem 5.3.4 : Let T be a surjective isometric multiplier on $L^p(G,A)$. Then T maps $L^1 \cap L^p(G,A)$ onto $L^1 \cap L^p(G,A)$ and preserves the L^1-norm.

Proof : Let $\sum_{i=1}^{n} a_i\chi_{C_i}$ be an A-valued simple function where C_i's are pairwise disjoint. Then by Lemma 5.3.3, for almost all $s \in G$ there exist disjoint measurable sets D_i with $\lambda(C_i) = \lambda(D_i)$ and elements $b_i(s) \in A$ with $\|b_i(s)\| = \|a_i\|$ such that

$$(T(a_i\,\chi_{C_i}))(s) = b_i(s)\,\chi_{D_i}(s)$$

Then

$$\|T\,(\sum_{i=1}^{n} a_i\,\chi_{C_i})\|_1 = \int_G \|\sum_{i=1}^{n} b_i(s)\,\chi_{D_i}(s)\|\,d\,\lambda(s)$$

$$= \sum_{i=1}^{n} \|a_i\|\,\|\chi_{D_i}\|_1$$

$$= \sum_{i=1}^{n} \|a_i\|\,\|\chi_{C_i}\|_1 \text{ (since } \lambda(C_i) = \lambda(D_i))$$

$$= \|\sum_{i=1}^{n} a_i\,\chi_{C_i}\|_1$$

Thus $\| T(\sum_{i=1}^{n} a_i \chi_{C_i}) \|_1 = \| \sum_{i=1}^{n} a_i \chi_{C_i} \|_1$.

Therefore T can be uniquely extended to $L^1(G,A)$ as an isometric multiplier of $L^1(G,A)$ Consequently T maps $L^1 \cap L^p(G,A)$ into $L^1 \cap L^p(G,A)$.

Replacing T by T^{-1}, we can similarly show that T^{-1} maps $L^1 \cap L^p(G,A)$ into $L^1 \cap L^p(G,A)$ Therefore it follows that T maps $L^1 \cap L^p(G,A)$ onto $L^1 \cap L^p(G,A)$ and preserves the L^1-norm.

This completes the proof.

As an application of Theorems 5.3.4 and 5.2.4, we obtain the following result.

Theorem 5.3.5 : Let G be a locally compact abelian group and A be a commutative semi-simple Banach algebra with a minimal approximate identity Suppose $1 < p < \infty$, $p \neq 2$ and $L^p(G,A)$ satisfies the property 'C'. Then every surjective isometric multiplier of $L^p(G,A)$ is of the form $F \delta_t$ where F is a surjective isometric multiplier of A and $t \in G$.

Proof : Let T be a surjective isometric multiplier of $L^p(G,A)$. By Theorem 5.3 4, T maps $L^1 \cap L^p(G,A)$ onto $L^1 \cap L^p(G,A)$ and preserves the L^1-norm. Therefore T can be uniquely extended to $L^1(G,A)$ as a surjective isometric multiplier of $L^1(G,A)$. It now follows from Theorem 5.2.4 that T has the required form.

§4. Isometric Isomorhisms

Throughout the section G_1, G_2 denote locally compact abelian groups with Haar measure λ_1, λ_2 respectively and A_1, A_2 be commutative semi-simple Banach algebras with identity such that $\ell_1^2 \notin A_1$.

In this section, we prove some results about isometric isomorphisms of $L^1(G_1, A_1)$ onto $L^1(G_2, A_2)$

We prove that if there exists an isometric isomorphism of $L^1(G_1, A_1)$ onto $L^1(G_2, A_2)$ then the groups G_1, G_2 are topologically isomorphic and the algebras $I'(A_1)$, $I'(A_2)$ are isometrically isomorphic. In the case when $I'(A_1) = A_1$, we give an explicit characterization of an isometric isomorphism of $L^1(G_1, A_1)$ onto $L^1(G_2, A_2)$

We also prove that every isometric isomorphism of $M(G_1, A_1)$ onto $M(G_2, A_2)$ maps $L^1(G_1, A_1)$ onto $L^1(G_2, A_2)$.

The main result of the section is the following Theorem.

Theorem 5.4.1 : Let ψ be an isometric isomorphism of $L^1(G_1, A_1)$ onto $L^1(G_2, A_2)$ Then there exists a topological isomorphism of G_1 onto G_2 and an isometric isomorphism of $I'(A_1)$ onto $I'(A_2)$.

Proof : Let e, e', e_1 and e_2 denote the identities of G_1, G_2, A_1 and A_2 respectively. ψ can be extended to $M(L^1(G_1, A_1)) \cong M(G_1, A_1)$ as an isometric isomorphism ψ_1 of $M(G_1, A_1)$ onto $M(G_2, A_2)$ defined by $\psi_1(T) = \psi\, T\, \psi^{-1}$ for all $T \in M(L^1(G_1, A_1))$.

Let $s \in G$ Then $\psi_1(e_1\ \delta_s)$ is an isometric multiplier of $L^1(G_1,A_1)$ Hence there exists an element $\phi(s) \in G_2$ and an element $\beta(s) \in I(A_2)$ such that $\psi_1(e_1\ \delta_s) = \beta(s)\ \delta_{\phi(s)}$. Thus we get mappings $\phi : G_1 \longrightarrow G_2$ and $\beta . G_1 \longrightarrow I(A_2)$ It is easy to see that ϕ is a homomorphism and $\beta(st) = \beta(s)\ \beta(t)$ for $s, t \in G_1$.

We now show that ϕ is continuous Suppose not Then there exists a net $\{s_\alpha\} \subseteq G_1$ such that $s_\alpha \longrightarrow e$ and a neighbourhood W of e' such that $\phi(s_\alpha) \notin W$ for each α. Let $f \in L^1(G_1,A_1)$. Then

$$\delta_{s_\alpha} * f \longrightarrow f \text{ in the norm of } L^1(G_1,A_1).$$

Since ψ is continuous, it follows that

$$\psi\ (\delta_{s_\alpha} * f) \longrightarrow \psi(f) \text{ in the norm of } L^1(G_2,A_2).$$

Now $\{\psi_1(e_1\ \delta_{s_\alpha})\} \subseteq M(G_2,A_2) \subseteq M(G_2,A_2^{**}) = (C_o(G_2,A_2^*))^*$. Therefore $\{\psi_1(e_1\delta_{s_\alpha})\}$ can be considered as a bounded net in $(C_o(G_2,A_2^*))^*$. Hence there exists a subnet of $\{\psi_1(e_1\delta_{s_\alpha})\}$ (which we continue to denote by $\{\psi_1(e_1\delta_{s_\alpha})\}$) and a measure $\mu \in M(G_2,A_2^{**})$ such that $\psi_1(e_1\delta_{s_\alpha})$ converges to μ in the weak* topology of $(C_o(G_2,A_2^*))^*$.

Let $g \in C_o(G_2,A_2^*)$ Then

$$\begin{aligned}\langle \psi f\ ,\ g\rangle &= \lim_\alpha \langle \psi(e_1\ \delta_{s_\alpha} * f)\ ,\ g\rangle \\ &= \lim_\alpha \langle \psi_1(e_1\ \delta_{s_\alpha}) * \psi f\ ,\ g\rangle \\ &= \lim_\alpha \langle \psi_1(e_1\ \delta_{s_\alpha})\ ,\ g * (\psi f)\tilde{}\rangle.\end{aligned}$$

Note that we have considered A_2^* and A_2^{**} as A_2-modules and consequently $g * (\psi f)\tilde{} \in C_o(G_2,A_2^*)$

Furthermore,

$$\lim_{\alpha} \langle \psi_1(e_1\ \delta_{s_\alpha})\ ,\ g * (\psi f)\tilde{}\rangle = \langle\ \mu\ ,\ g * (\psi f)\tilde{}\rangle$$

$$= \langle\ \mu * \psi f\ ,\ g\rangle .$$

Therefore $\psi_1(e_1\ \delta_{s_\alpha} * f)$ converges to $\mu * \psi f$ in the weak* topology of $(C_o(G_2, A_2^*))^*$. Also $\psi_1(e_1\ \delta_{s_\alpha} * f)$ converges in norm to ψf. Therefore,

$$\mu * \psi f = \psi f \text{ for all } f \in L^1(G_1, A_1)$$

Since ψ is onto, it follows that

$$\mu * g = g \text{ for all } g \in L^1(G_2, A_2).$$

Therefore $\mu = e_2\ \delta_{e'}$.

Since $\psi_1(e_1\delta_{s_\alpha}) = \beta(s_\alpha)\delta_{\phi(s_\alpha)}$, it follows that $\beta(s_\alpha)\ \delta_{\phi(s_\alpha)}$ converges to $e_2\delta_{e'}$ in the weak* topology of $(C_o(G_2, A_2^*))^*$. Now we choose a function $f \in C_o(G_2)$ such that $f(e') = 1$ and Supp. $f \subseteq W$. Also choose an element $a \in A_2^*$ such that $a^*(e_2) \neq 0$. Then

$$f(\phi(s_\alpha))\ a^*(\ \beta(s_\alpha)) \longrightarrow a^*(e_2)f(e').$$

But by our choice of s_α, $\phi(s_\alpha) \notin W$ for each α. Since Supp. $f \subseteq W$, it follows that $f(\phi(s_\alpha)) = 0$ for each α. But $a^*(e_2)f(e') \neq 0$. This contradiction proves that ϕ is continuous.

We now show that ϕ is bijective. First, we prove that if $a_1 \in I(A_1)$ then there exists an element $a_2 \in I(A_2)$ such that $\psi_1(a_1\ \delta_e) = a_2\ \delta_{e'}$

Since $\psi_1(a_1\ \delta_e)$ is an isometric multiplier of $L^1(G_2, A_2)$, there exist $a_2 \in I(A_2)$ and $t \in G_2$ such that $\psi_1(a_1\ \delta_e) = a_2\ \delta_t$. We claim

that $t = e'$. Suppose $t \neq e'$.

Since $\psi_1(e_1\ \delta_e) = e_2\ \delta_{e'}$, therefore for $\lambda, \mu \in \mathbb{C}$,

$$\psi_1(\lambda a_1 + \mu e_1)\delta_e) = \lambda\ a_2\ \delta_t + \mu\ e_2\ \delta_{e'}$$

Since ψ_1 is an isometry, it follows that

$$\|\lambda a_1 + \mu e_1\| = |\lambda| + |\mu|.$$

Therefore the space generated by $\{a_1, e_1\}$ is isometrically isomorphic to ℓ_1^2. This contradicts that $\ell_1^2 \not\in A_1$. Therefore $t = e'$

Replacing ψ_1 by ψ_1^{-1} and using the fact that $\ell_1^2 \not\in A_2$, we can show that if $a_2 \in I(A_2)$ then there exists an element $a_2 \in I(A_1)$ such that $\psi_1^{-1}(a_2\ \delta_{e'}) = a_1\ \delta_e$.

To show the injectivity of ϕ, we must show that if $s \in G$ such that $\phi(s) = e'$ then $s = e$.

Now

$$\psi_1(e_1\ \delta_s) = \beta(s)\ \delta_{e'}\ .$$

Therefore $(\beta(s))^{-1}\ \delta_{e'} * \psi_1(e_1\ \delta_s) = e_2\ \delta_{e'}$.

Let $a \in I(A_1)$ be such that $\psi_1^{-1}((\ \beta(s))^{-1}\ \delta_{e'}) = a\ \delta_e$

Then $\psi_1(a\ \delta_e) * \psi_1(e_1\ \delta_s) = e_2\ \delta_{e'}$.

Therefore $\psi_1(a\ \delta_s) = e_2\ \delta_{e'}$.

Since ψ_1 is an isomorphism, it follows that $a = e_1$ and $s = e$. Suppose $t \in G_2$ Let $\beta'(t) \in I(A_1)$ and $\phi'(t) \in G_1$ be such that

$$\psi_1^{-1}(e_2\ \delta_t) = \beta'(t)\ \delta_{\phi'(t)} \tag{4.1}$$

We prove that ϕ is surjective by showing that $\phi\ (\phi'(t)) = t$.

From (4.1), we have

$$(\beta'(t))^{-1}\ \delta_e * \psi_1^{-1}(e_2\ \delta_t) = e_1\ \delta_{\phi'(t)}$$

Let $a' \in I(A_2)$ be such that $\psi_1((\beta'(t))^{-1}\ \delta_e) = a'\delta_{e'}$.

Then

$$\psi_1^{-1}(a'\delta_{e'}) * \psi_1^{-1}(e_2\ \delta_t) = e_1\ \delta_{\phi'(t)}$$

Thus

$$\psi_1^{-1}(a'\delta_t) = e_1\ \delta_{\phi'(t)}$$

Therefore $\psi_1(e_1\ \delta_{\phi'(t)}) = a'\delta_t$.

Hence $\phi(\phi'(t)) = t$.

Replacing ψ_1 by ψ_1^{-1}, we can show that ϕ^{-1} is continuous.

Thus we have proved that ϕ is a topological isomorphism of G_1 onto G_2.

Suppose $a \in I'(A_1)$. Let $\{a_n\}$ be a sequence in the linear span of $I(A_1)$ such that $a_n \longrightarrow a$. Let $\{b_n\}$ be the sequence in the linear span of $I(A_2)$ such that $\psi_1(a_n\delta_e) = b_n\delta_{e'}$. Since ψ_1 is continuous, it follows that $\psi_1(a_n\delta_e) = b_n\delta_{e'} \longrightarrow \psi_1(a\ \delta_e)$.

Since ψ_1 is an isometry, it follows that $\|a_n - a_m\| = \|b_n - b_m\| \longrightarrow 0$ as $n,m \longrightarrow \infty$. Hence there exists an element $b \in I'(A_2)$ such that $b_n \longrightarrow b$. Therefore, we get that $\psi_1(a\ \delta_e) = b\ \delta_{e'}$.

Thus we have proved that if $a \in I'(A_1)$, then there exists an element $b \in I'(A_2)$ such that $\psi_1(a\delta_e) = b\delta_{e'}$.

We define $F : I'(A_1) \longrightarrow I'(A_2)$ by

$$F(a) = b \text{ where } \psi_1(a\ \delta_e) = b\ \delta_{e'}.$$

It is easy to see that F is an isometric isomorhism of $I'(A_1)$ onto $I'(A_2)$.

This completes the proof of the Theorem.

Remarks 5.4.2 (1) In Theorem 5.4.1, the algebras A_i's are such that $\ell_1^2 \not\subseteq A_i$. ℓ_∞ and C(G), for a compact space G are commutative Banach algebras with identity which do not ℓ_1^2. Note that no strictly convex space can contain ℓ_1^2

(2) The proof of Theorem 5.4 1 holds even if we replace $L^1(G_i,A_i)$ by vector valued Segal algebras $S(G_i,A_i)$ whose isometric multipliers are of the form $a_i\ \delta_{t_i}$ where $a_i \in I(A_i)$ and $t_i \in G_i$.

It follows from the results of section 2 that if M (S(G,A)) is isomorphic to M(G,A) then isometric multipliers of S(G,A) are of the form $a\ \delta_t$ where $a \in I(A)$ and $t \in G$.

If G is a non-compact locally compact abelian group and A is a commutative semi-simple Banach algebra with identity then it is shown in [50] that the multiplier space of the following Segal algebras is isomorphic to M(G,A).

(i) $L^1 \cap C_o(G,A)$

(ii) $L^1 \cap L^p(G,A)$, $1 < p < \infty$

(iii) $A_p(G,A) = \{f \in L^1(G,A) : \hat{f} \in L^p(\Gamma,A)\}$, $1 \le p < \infty$.

Therefore Theorem 5.4.1 is applicable to all the cases mentioned in (i), (ii) and (iii).

We now give some examples of commutative Banach algebras with identity for which $I'(A) = A$ and also others for which $I'(A) \subsetneq A$.

Examples:(1) Let A be the algebra $\ell_1(\mathbb{Z})$ with multiplication defined by convolution We know [51] that isometric multipliers of A are given by $c\,\chi_{\{k\}}$ where $k \in \mathbb{Z}$ and c is a constant with $|c| = 1$. The linear span of I(A) which is the set of sequences having finitely many non-zero entries, is dense in $\ell_1(\mathbb{Z})$ Hence in this case $I'(A)=A$.

(2) Let G be a compact Hausdorff space Let A = C(G). We consider pointwise multiplication in C(G). It is known that the isometric multipliers of A are given by continuous functions on G having absolute value one. Thus $I'(A)$ is a closed subalgebra of A which contains conjugate of its functions, the constant function 1 and separates points. Hence $I'(A) = A$.

(3) Let G be a non-discrete locally compact abelian group. Let A = M(G) with multiplication defined by convolution. Then it can be shown as in the proof of Theorem 2 of [31] that the isometric multipliers of M(G) are of the form $c\,\delta_t$ where c is a constant with $|c| = 1$ and $t \in G$. $I'(A)$ is the set of all discrete measures. Hence $I'(A) \neq A$.

We have seen in Theorem 5.4.1 that if there is an isometric isomorphism ψ of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ then ψ can be extended to $M(G_1,A_1)$ as an isometric isomorphism ψ_1 of $M(G_1,A_1)$ onto $M(G_2,A_2)$ defined by $\psi_1(T) = \psi\, T\, \psi^{-1}$ for $T \in M(L^1(G_1,A_1))$ We also get the mappings $\phi : G_1 \longrightarrow G_2$ and $\beta \cdot G_1 \longrightarrow I(A_2)$ such that $\psi_1(e_1\delta_s) = \beta(s)\,\delta_{\phi(s)}$. We shall prove that every isometric isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$ maps $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$. This implies that there exists a one-to-one correspondence between isometric isomorphisms of $L^1(G_2,A_2)$ onto

$L^1(G_2,A_2)$ and isometric isomorphisms of $M(G_1,A_1)$ onto $M(G_2,A_2)$. To prove this result, we first prove the following Lemma.

Lemma 5.4.3 · The mapping $s \longrightarrow \beta(s)$ from G_1 to $I(A_2)$ described above is continuous.

Proof : Since ψ_1 is continuous, the mapping $s \longrightarrow \psi_1(e_1\delta_s)$ is continuous from G_1 to the space $M(L^1(G_2,A_2))$ in the strong operator topology Also since ϕ is continuous, the mapping $s \longrightarrow \delta_{\phi(s^{-1})}$ is continuous from G_1 to $M(L^1(G_2,A_2))$ in the strong operator topology. Therefore the mapping

$$s \longrightarrow \psi_1(e_1\,\delta_s) * e_2\,\delta_{\phi(s^{-1})}$$

is continuous from G_1 to $M(L^1(G_2,A_2)$ in the strong operator topology.

Since $\psi_1(e_1\,\delta_s) * e_2\,\delta_{\phi(s^{-1})} = \beta(s)\,\delta_{\phi(s)} * e_2\,\delta_{\phi(s^{-1})}$

$$= \beta(s)\,\delta_{e'}.$$

Therefore the mapping $s \longrightarrow \beta(s)\,\delta_{e'}$ is continuous from G_1 to $M(L^1(G_2,A_2))$ in the strong operator topology.

Let $s,\ t \in G_1$ and $g \in L^1(G_2)$. Then

$$\|\beta(s)\,e_2\,g - \beta(t)\,e_2\,g\|_1 \longrightarrow 0 \text{ as } s \longrightarrow t.$$

Therefore for $\gamma \in \Gamma_2$

$$\beta(s)\,\hat{g}(\gamma) \longrightarrow \beta(t)\,\hat{g}(\gamma) \text{ as } s \longrightarrow t$$

This proves continuity of the mapping β.

We now prove the following Theorem

Theorem 5 4.4: Let ψ be an isometric isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$ Then $\psi\,(L^1(G_1,A_1)) = L^1(G_2,A_2)$

Proof : Let e, e', e_1 and e_2 denote the identities of G_1, G_2, A_1 and A_2 respectively Since ψ is an isometric isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$, therefore as in the proof of Theorem 5.4 1 and Lemma 5.4.3 there exists a topological isomorphism ϕ of G_1 onto G_2 and a continuous mapping β of G_1 into $I(A_2)$ such that for $s \in G_1$

$$\psi(e_1\ \delta_s) = \beta(s)\ \delta_{\phi(s)}.$$

Similarly by considering ψ^{-1} we shall get a continuous mapping β' of G_2 into $I(A_1)$ such that for $t \in G_2$

$$\psi^{-1}(e_2\ \delta_t) = \beta'(t)\ \delta_{\phi^{-1}(t)}.$$

Next we show that if $f \in L^1(G_1,A_1)$ then the map $t \longrightarrow \tau_t(\psi f)$ is continuous from G_2 to $M(G_2,A_2)$.

Since $\tau_t(\psi f) = e_2\ \delta_t * \psi f$

and $\psi^{-1}(e_2\ \delta_t) = \beta'(t)\ \delta_{\phi^{-1}(t)}$,

therefore

$$\tau_t(\psi f) = \psi\left(\beta'(t)\ \delta_{\phi^{-1}(t)}\right) * \psi f$$

$$= \psi\left(\beta'(t)\ \delta_{\phi^{-1}(t)} * f\right)$$

$$= \psi\left(\beta'(t)\ \tau_{\phi^{-1}(t)} f\right)$$

Hence $\|\tau_t(\psi f) - \psi f\| = \|\psi\left(\beta'(t)\ (\tau_{\phi^{-1}(t)} f)\right) - \psi f\|$

$$= \|\psi\left(\beta'(t)\,(\tau_{\phi^{-1}(t)}f) - f\right)\|$$

$$= \|\beta'(t)\,(\tau_{\phi^{-1}(t)}f) - f\|_1$$

$$\leq \|\beta'(t)\,\tau_{\phi^{-1}(t)}f - \tau_{\phi^{-1}(t)}f\|_1 + \|\tau_{\phi^{-1}(t)}f - f\|_1$$

$$= \|\beta'(t) - e_2\|\,\|f\|_1 + \|\tau_{\phi^{-1}(t)}f - f\|_1$$

which tends to zero as $t \longrightarrow e'$ because β and ϕ^{-1} are continuous and translation is continuous in $L^1(G_2,A_2)$.

It now follows from Theorem 4 of [45] that $\psi f \in L^1(G_2,A_2)$ Similarly we can show that $\psi^{-1}g \in L^1(G_1,A_1)$ for $g \in L^1(G_2,A_2)$.

Hence $\psi\,(L^1(G_1,A_1)) = L^1(G_2,A_2)$.

Using the techniques similar to those in Wendel [51], in the following theorem, we describe the nature of an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$.

Theorem 5.4.5 : Let ϕ be a topological isomorphism of G_1 onto G_2, F be an isometric isomorphism of A_1 onto A_2 and β be a continuous mapping of G_1 into $I(A_2)$ with $\beta(st) = \beta(s)\,\beta(t)$ for $s,t \in G_1$. Suppose c is the unique positive constant such that $\lambda_1(E) = c\,\lambda_2(\phi(E))$ for all Borel measurable subsets E of G_1. Then U, defined by $(Uf)(\phi(s)) = c\,\beta(s)\,F(f(s))$ a.e. for $f \in L^1(G_1,A_1)$ and $s \in G_1$, is an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$. Furthermore, if $I'(A_1) = A_1$ then any isometric isomorphism U of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ is given in this form.

Proof : Let $f, g \in L^1(G_1,A_1)$ and $s \in G_1$. It is easy to see that U is linear on $L^1(G_1,A_1)$. We now show that $U(f*g) = (Uf)*(Ug)$.

By definition,

$$(U(f*g))(\phi(s)) = c\,\beta(s)\,F((f*g)(s)) \text{ a.e.}$$

$$= c\,\beta(s)\,F\left(\int_{G_1} f(st)g(t^{-1})d\,\lambda_1(t)\right)$$

$$= c\,\beta(s)\int_{G_1} F\,(f(st))\,F\,(g(t^{-1}))d\,\lambda_1(t)$$

$$= c^{-1}\int_{G_1}\left[c\,\beta(st)\,F(f(st))\right]\left[c\,\beta(t^{-1})F(g(t^{-1}))\right]d\,\lambda_1(t)$$

$$= c^{-1}\int_{G_1} Uf\,(\phi(st))\,Ug\,((\phi(t^{-1}))d\,\lambda_1(t)$$

$$= \int_{G_2} Uf\,(\phi(s)\phi(t))\,Ug\,((\phi(t))^{-1})d\,\lambda_2(\phi(t))$$

(since $\lambda_1(E) = c\,\lambda_2(\phi(E))$)

$$= ((Uf) * (Ug))(\phi(s)).$$

Furthermore, $\|Uf\|_1 = \|f\|_1$.

Next we show that U is onto. Let $h \in L^1(G_2,A_2)$. We define a function $f \in L^1(G_1,A_1)$ by

$$f(s) = c^{-1}\,F^{-1}\left(\beta(s^{-1})\,h(\phi(s))\right) \text{ for almost all } s \in G_1.$$

It can be easily seen that $f \in L^1(G_1,A_1)$ and $Uf = h$. Therefore U is an isometric isomorphism of $L^1(G_2,A_2)$ such that for $h \in L^1(G_2,A_2)$ and $s \in G_1$, $(U^{-1}h)(s) = c^{-1}\,F^{-1}\left(\beta(s^{-1})\,h(\phi(s))\right)$ a.e.

To prove the second assertion, let $I'(A_1) = A_1$ and ψ be an isometric isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$. Then we know from the proof of Theorem 5.4.1 that there exists a topological

isomorphism ϕ of G_1 onto G_2, an isometric isomorphism F of A_1 onto A_2 and a continuous map β of G_1 into $I(A_2)$ with $\beta(st) = \beta(s)\,\beta(t)$ for $s, t \in G_1$ such that for $a \in A_1$,

$$\left.\begin{aligned} \psi\, \tau_t\, \psi^{-1} &= \beta(t)\, \tau_{\phi(t)} \\ \text{and} \quad \psi\, a\, \psi^{-1} &= F(a)\, I\ , \end{aligned}\right] \qquad (4\ 2)$$

where I is the identity operator on $L^1(G_2, A_2)$.

Let c be the unique positive constant such that $\lambda_1(E) = c\lambda_2(\phi(E))$ for all $E \in \Sigma$ Define U by $(Uf)(\phi(s)) = c\beta(s)\, F(f(s))$ a.e for all $f \in L^1(G_1, A_1)$ and $s \in G_1$.

Then for each $h \in L^1(G_2, A_2)$,

$$\begin{aligned} (U\, \tau_t\, U^{-1}h)\,(\phi(s)) &= U\left(\tau_t\, U^{-1}h\right)(\phi(s)) \\ &= c\, \beta(s)\, F\left((\tau_t\, U^{-1}h)\,(s)\right) \\ &= c\, \beta(s)\, F\left((U^{-1}h)\,(t^{-1}s)\right) \\ &= c\, \beta(s)\, F\left(c^{-1}\, F^{-1}\left(\beta(s^{-1}t)\, h(\phi(t^{-1}s))\right)\right) \\ &= \beta(t)\, h(\phi(t^{-1})\, \phi(s)) \\ &= \beta(t)\, (\tau_{\phi(t)}h)\,(\phi(s)). \end{aligned}$$

Thus we have

$$U\, \tau_t\, U^{-1} = \beta(t)\, \tau_{\phi(t)}, \quad \text{for } t \in G_1.$$

Also for $a \in A_1$,

$$\begin{aligned} (U\, a\, U^{-1}h)\,(\phi(s)) &= U(a\, U^{-1}h)\,(\phi(s)) \\ &= c\, \beta(s)\, F\left((a\, U^{-1}h)(s)\right) \end{aligned}$$

$$= c\,\beta(s)\,F\left(a\,c^{-1}\,F^{-1}\left(\beta(s^{-1})h(\phi(s))\right)\right)$$

$$= F(a)\,h(\phi(s)).$$

Therefore $U\,a\,U^{-1} = F(a)\,I$

and $U\,\tau_t\,U^{-1} = \beta(t)\,\tau_{\phi(t)}$

From (4.2), it follows that for $t \in G_1$ and $a \in A_1$, we have

$$U\,\tau_t\,U^{-1} = \psi\,\tau_t\,\psi^{-1}$$

and $U\,a\,U^{-1} = \psi\,a\,\psi^{-1}$

Therefore $\psi^{-1}\,U\,\tau_t = \tau_t\,\psi^{-1}\,U$

and $\psi^{-1}\,U\,a = a\,\psi^{-1}\,U.$

Let $S = \psi^{-1}U$ Then $S\,\tau_t = \tau_t S$ and $Sa = aS$

It follows from Proposition 5.3.2 that S is a multiplier on $L^1(G_1,A_1)$.

Therefore for $f, g \in L^1(G_1,A_1)$

$$S(f * g) = f * Sg.$$

Also S is an isomorphism of $L^1(G_1,A_1)$ onto itself, therefore

$$S(f * g) = Sf * Sg.$$

Thus we have

$$Sf * Sg = f * Sg.$$

Since S is onto, it follows that

$$Sf * k = f * k \text{ for } f, k \in L^1(G_1,A_1).$$

Therefore $Sf = f$ for all $f \in L^1(G_1,A_1)$.

Hence $S = \psi^{-1}U$ is the identity operator on $L^1(G_1,A_1)$. This proves that $U = \psi$. This completes the proof of the theorem.

CHAPTER VI

BIPOSITIVE ISOMORPHISMS

§1. INTRODUCTION

We begin with some definitions

Let G, G_1, G_2 denote locally compact groups.

Definition 6.1.1 : A mapping $T : L^p(G_1) \longrightarrow L^p(G_2)$ is called *positive* if for $f \in L^p(G_1)$ and $f \geq 0$ almost everywhere, we have $Tf \geq 0$ almost everywhere.

Definition 6.1.2 : A multiplier of $L^p(G)$ is called a *positive multiplier* if it is a positive operator.

Definition 6.1.3 : A mapping $\psi : M(L^p(G_1)) \longrightarrow M(L^p(G_2))$ is called *bipositive* if $\psi(T)$ is a positive multiplier if and only if T is a positive multiplier of $L^p(G_1)$. If in addition, ψ is an isomorphism then ψ is called a *bipositive isomorphism*.

Kawada [20] initiated the study of bipositive isomorphisms. He proved that if there exists a bipositive isomorphism of $L^1(G_1)$ onto $L^1(G_2)$ then G_1 and G_2 are topologically isomorphic. Edwards [12] showed that if G_1, G_2 are compact groups and if there exists a bipositive isomorphism of $L^p(G_1)$ onto $L^p(G_2)$ $(1 \leq p < \infty)$ then G_1 and G_2 are topologically isomorphic. Later, it was established by Gaudry [14] that the groups G_1, G_2 are topologically isomorphic if there is a bipositive isomorphism of $M(L^p(G_1))$ onto $M(L^p(G_2))$.

In each of these results, the characterization of positive multipliers of $L^p(G)$ plays a significant role. Brainerd and Edwards [2] proved that if G is a locally compact abelian group then the positive multipliers of $L^p(G)$ are given by positive measures in M(G). This result for a compact group G was proved by Edwards [12]

In order to give meaning to the concept of positivity in the vector valued case, we shall deal with Banach lattice algebra valued function spaces

For the definitions and results about Banach lattice algebras, we refer to Schaefer [38].

Definition 6.1.4 : Let A be a real vector lattice. A norm $\|\cdot\|$ on A is called a *lattice norm* if $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$ for all $x,y \in A$ ($|x|$ denotes the modulus of x). If in addition, $(A, \|\cdot\|)$ is complete, then A is called a *Banach lattice* By A^+ we denote the set of all positive element of A.

Definition 6.1.5 : A real vector lattice A is said to be ℓ^1-*relatively complete* if $0 \leq x_n \leq \lambda_n x$ where x_n, $x \in A$ and $\{\lambda_n\} \in \ell^1$, implies the order convergence of $\sum_1^\infty x_n$.

Definition 6.1.6 : Let A be a real ℓ^1-relatively complete vector lattice. Let $A_{\mathbb{C}} = A + iA$ denote the complexification of the vector space A. Then for $z = x + iy \in A_{\mathbb{C}}$, the supremum

$$|z| = \sup_{0 \leq \theta \leq 2\pi} |(\cos\theta)x + (\sin\theta)y| \qquad (1.1)$$

exists in A. Moreover, the following properties of modulus function $z \longrightarrow |z|$ are easy to verify.

(i) $|z| = 0$ if and only if $z = 0$.

(ii) $|\alpha z| = |\alpha|\,|z|$ for all $\alpha \in \mathbb{C}$ and $z \in A_{\mathbb{C}}$.

(iii) $|z_1+z_2| \leq |z_1| + |z_2|$ for all $z_1, z_2 \in A_{\mathbb{C}}$.

The complexification of the vector lattice A endowed with the modulus function is called a *complex vector lattice.*

If A is a normed lattice then we define $\|\cdot\|$ on $A_{\mathbb{C}}$ by

$$\|z\| = \| |z| \| \text{ for } z \in A_{\mathbb{C}} \tag{1.2}$$

It is easy to see that if A is complete then $A_{\mathbb{C}}$ is also complete. By a *complex Banach lattice*, we mean the complexification $A_{\mathbb{C}}$ of a Banach lattice A endowed with the modulus function (1.1) and the norm (1.2).

Definition 6.1.7 : Let A be a Banach algebra and a Banach lattice such that $|xy| \leq |x|\,|y|$ for all $x, y \in A$ then A is called a *Banach lattice algebra.*

Definition 6.1.8 : Let A be a Banach lattice algebra An element $e \in A^+$ is called *identity* of the Banach lattice algebra A if $ex=x$ for all $x \in A$.

It can be checked that if A is a Banach lattice algebra then its complexification $A_{\mathbb{C}}$ becomes a Banach algebra and is called a *complex Banach lattice algebra.*

We give below some examples of Banach lattice algebras.

(1) Let G be a topological space. Let $C^b(G)$ denote the space of all bounded real valued continuous functions on G. With pointwise multiplication, supremum norm and the usual ordering, $C^b(G)$ becomes a Banach lattice algebra with

identity

(2) Let G be a locally compact group. Let $L^1_{\mathbb{R}}(G)$ denote the space of real valued functions integrable with respect to the Haar measure on G. Then $L^1_{\mathbb{R}}(G)$ becomes a Banach lattice algebra with convolution as multiplication and the usual ordering

(3) [38, P 297] Let X be a normed vector lattice Let $\mathfrak{L}^r(X)$ denote the vector space of linear operators on X possessing a decomposition $T = T_1 - T_2$ where T_1, T_2 are positive and continuous $\mathfrak{L}^r(X)$ becomes a Banach lattice algebra with the usual ordering and the norm

$$\|T\|_r = \inf \|T_1 + T_2\|$$

where the infimum is taken over all such decompositions

Let A, A_1, A_2 denote Banach lattice algebras with identity. A function $f \in L^1(G,A)$ is said to be *positive* if $f(s) \in A^+$ for almost all $s \in G$. A multiplier T of $L^1(G,A)$ is said to be *positive* if Tf is positive whenever f is positive. A measure $\mu \in M(G,A)$ is called *positive* if it takes its values in A^+.

In Section 2, we characterize positive multipliers of $L^1(G,A)$. Infact, we prove that positive multipliers of $L^1(G,A)$ are given by positive measures in M(G,A).

Let the algebras A_1's be such that inverse of each positive invertible element of A_1 is positive. Let $J^+(A_1)$ denote the set of all positive invertible elements of A_1. In Section 3, we prove that if there is a continuous bipositive isomorphism of $L^1(G_1,A_1)$

onto $L^1(G_2,A_2)$ then there exists a topological isomorphism of G_1 onto G_2 and a continuous bipositive isomorphism of A_1 onto A_2. As in the isometric case, we give a characterization of continuous bipositive isomorphisms of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ Using similar techniques as in Theorem 5.4 4, we prove that any continuous bipositive isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$ maps $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$.

§2. Positive Multipliers

Throughout the section, G denotes a locally compact group and A denote a Banach lattice algebra with identity.

In the following Theorem, we determine the positive multipliers of $L^1(G,A)$.

Theorem 6.2.1 : The positive multipliers of $L^1(G,A)$ can be identified with the positive measures belonging to $M(G,A)$.

Proof : Suppose T is a positive multiplier on $L^1(G,A)$. Let μ be the measure in $M(G,A)$ corresponding to T. Let $\{g_\alpha\}$ be an approximate identity of $L^1(G)$ consisting of positive elements. Suppose e denotes the identity of A. Then $\{f_\alpha = eg_\alpha\}$ becomes an approximate identity of $L^1(G,A)$ consisting of positive elements. Let $f \in C_c^+(G)$. Since $L^1(G) * C_o(G) = C_o(G)$, there exist functions $k \in L^1(G)$ and $h \in C_o(G)$ such that $f = k * h$.

Since $k * \mu \in L^1(G,A)$, it follows that

$$f_\alpha * k * \mu \longrightarrow k * \mu \text{ in the norm of } L^1(G,A).$$

Since $M(G,A^{**}) = (C_0(G,A^*))^*$, if $g \in C_0(G,A^*)$ and e' is the identity of G, then

$$(f_\alpha * k * \mu * g)(e') \longrightarrow (k * \mu * g)(e').$$

For $s \in G$, we have $f * \tau_s g = \tau_s(f * g)$ Therefore

$$(f_\alpha * k * \mu * g)(s) \longrightarrow (k * \mu * g)(s) \text{ for all } g \in C_0(G,A^*).$$

Let $a^* \in A^*$ and $g = a^*h$. Then

$$a^*\Big[(f_\alpha * k * \mu * h)(s)\Big] \longrightarrow a^*\Big[(k * \mu * h)(s)\Big].$$

Therefore

$$a^*\Big[(f_\alpha * f * \mu)(s)\Big] \longrightarrow a^*\Big[(f * \mu)(s)\Big]$$

Since $f_\alpha * f \in L^1(G,A)$ and $T(f_\alpha * f) = f_\alpha * f * \mu$, it follows that

$$a^*\Big[(T(f_\alpha * f))(s)\Big] \longrightarrow a^*\Big[(f * \mu)(s)\Big].$$

We note that for each α, $f_\alpha * f$ is positive almost everywhere. Since T is positive, it follows that $T(f_\alpha * f)$ is positive almost everywhere. Further, $T(f_\alpha * f) = Tf_\alpha*(fe)$ is continuous and A^+ is closed, therefore $(T(f_\alpha * f))(s) \in A^+$ for all $s \in G$. Thus $\{(T(f_\alpha * f))(s)\}$ is a net of positive elements of A which converges weakly to $(f * \mu)(s)$ for each $s \in G$. It follows from Corollary [38, P 89] that for each $s \in G$, $(T(f_\alpha * f))(s)$ converges to $(f * \mu)(s)$ in the norm of A. Since $(T(f_\alpha * f))(s) \in A^+$ and A^+ is closed, $(f * \mu)(s) \in A^+$. Thus we have proved that for each $f \in C_c^+(G)$ and $s \in G$, $(f * \mu)(s) \in A^+$. Therefore

$(\tilde{f} * \mu)(e') = \int f \, d\mu \in A^+$ for each $f \in C_c^+(G)$.

We now show that for each Borel measurable set E, $\mu(E) \in A^+$. Let $\{f_n\}$ be a sequence of functions in $C_c^+(G)$ such that $f_n \longrightarrow \chi_E$ in $L^1(|\mu|)$. Then $\int f_n \, d\mu \longrightarrow \int \chi_E \, d\mu = \mu(E)$. Therefore $\mu(E) \in A^+$.

Hence μ is a positive measure.

Conversely, suppose μ is a positive measure in $M(G,A)$. Let $Tf = f * \mu$ for all $f \in L^1(G,A)$. It can be easily seen that T defines a positive multiplier on $L^1(G,A)$.

Remark 6.2.2 : Let G be a locally compact abelian group and A be a commutative Banach lattice albegra with identity If $S(G,A)$ denotes a Segal algebra consisting of vector valued functions whose multiplier space is isomorphic to $M(G,A)$, then we can show that the positive multipliers of $S(G,A)$ are given by positive measures in $M(G,A)$. Infact, the arguments of the proof of Theorem 6 2.1 go through once we observe that $S(G,A)$ contains an approximate identity consisting of positive elements. This can be seen from the following Proposition.

Proposition 6.2.3 : Let $f \in L^1(G,A)$ be such that $\hat{f}$ has compact support. Then $f \in S(G,A)$. Furthermore $S(G,A)$ contains an approximate identity consisting of positive elements.

Proof : Since A contains the identity, $\Delta(A)$ is compact. Therefore $\mathcal{F}f$ has compact support. It follows from 5.2 7 (4) that $f \in S(G,A)$

Next we show that $S(G,A)$ contains an approximate identity consisting of positive elements. Suppose $\{g_\alpha\}$ is an approximate identity of $L^1(G)$ such that $g_\alpha \geq 0$ almost everywhere and $\hat{g}_\alpha$ has compact support for each α (See, Theorem 33.12 of [16]). Let e denote the identity of A. Then $\{f_\alpha = eg_\alpha\}$ is an approximate identity of $S(G,A)$ such that $\hat{f}_\alpha(s) \in A^+$ for almost all $s \in G$.

§3. Bipositive Isomorphisms

Throughout the section G_1, G_2 denote locally compact groups and A_1, A_2 denote Banach lattice algebras with identity such that inverse of each positive invertible element is positive. In this section, we prove some results about bipositive isomorphisms Following is the main result of this section

Theorem 6.3.1 : Suppose ψ is a continuous bipositive isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ Then there exists a topological isomorphism of G_1 onto G_2 and a continuous bipositive isomorphism of A_1 onto A_2.

Proof : Let e, e', e_1 and e_2 denote the identities of G_1, G_2, A_1 and A_2 respectively. ψ can be extended to $M(L^1(G_1,A_1)) \cong M(G_1,A_1)$ as a continuous bipositive isomorphism ψ_1 of $M(G_1,A_1)$ onto $M(G_2,A_2)$ defined by $\psi_1(T) = \psi\, T\, \psi^{-1}$ for $T \in M(L^1(G_1,A_1))$. Note that ψ^{-1} is continuous by Open mapping theorem. Since positive measures in $M(G_1,A_1)$ are identified with positive multipliers of $L^1(G_1,A_1)$, it follows that ψ_1 maps positive measures of $M(G_1,A_1)$ to positive measures of $M(G_2,A_2)$.

Next we show that if x is a positive invertible element of A_1 and $s \in G_1$ then $\psi_1(x\,\delta_s)$ has one point support

Suppose $\psi_1(x\,\delta_s) = \mu$. Then $\psi_1(x^{-1}\delta_{-s}) = \mu^{-1}$.

Since x and x^{-1} are positive elements of A_1 and ψ_1 is positive, it follows that μ and μ^{-1} are positive measures.

Let s_1, $s_2 \in$ Supp. μ, $s_1 \neq s_2$ and $t \in$ Supp. μ^{-1}. We choose a relatively compact neighbourhood U of e' such that

$$\left((s_1+U) + (t+U)\right) \cap \left((s_2+U) + (t+U)\right) = \phi.$$

Choose $\phi' \in C_c(G_2)$ such that $0 \leq \phi' \leq 1$ and Supp. $\phi' \subseteq U$

Define $\mu_1 = (\tau_{s_1} \phi')\ \mu + (\tau_{s_2} \phi')\mu$

and $\nu_1 = (\tau_t \phi')\mu^{-1}$

Then μ_1 and ν_1 are positive measures. Furthermore $\mu_1 \leq \mu$ and $\nu_1 \leq \mu^{-1}$. Therefore $\mu_1 * \nu_1 \leq \mu * \mu^{-1} = e_2\ \delta_{e'}$. But $\mu_1 * \nu_1$ is a positive measure with atleast two distinct points s_1+t and s_2+t in its support. This contradiction proves that μ has one point support

Let $s \in G_1$. Then there exist elements $\phi(s) \in G_2$ and $\beta(s) \in J^+(A_2)$ such that $\psi_1(e_1\ \delta_s) = \beta(s)\ \delta_{\phi(s)}$. Thus we get mappings ϕ of G_1 into G_2 and β of G_1 into $J^+(A_2)$ It can be easily seen that ϕ is a homomorphism. Also β is bounded and $\beta(st) = \beta(s)\ \beta(t)$ for $s, t \in G_1$. Using the arguments similar to the proof of continuity of ϕ in Theorem 5.4.1, we can show that the mapping ϕ is continuous

Next we show that if $x \in A_1$ then there exists a $y \in A_2$ such that $\psi_1(x\ \delta_e) = y\ \delta_{e'}$.

First, suppose x is a positive invertible element of A_1. Then as seen above there exists a positive invertible element y of A_2 and an element $t \in G_2$ such that $\psi_1(x\ \delta_e) = y\ \delta_t$.

Let $x_1 = \frac{x}{1+\|x\|} + e_1$. Then x_1 is invertible because

$$\|x_1 - e_1\| = \left\|\frac{x}{1+\|x\|} + e_1 - e_1\right\| < 1.$$

Thus x_1 is a positive invertible element of A_1 Therefore there exists a positive invertible element y_1 of A_2 and $t_1 \in G_2$ such that $\psi_1(x_1\ \delta_e) = y_1\ \delta_{t_1}$.

Furthermore,

$$\psi_1(x_1\ \delta_e) = \psi_1\left(\left(\frac{x}{1+\|x\|} + e_1\right)\ \delta_e\right)$$

$$= \frac{1}{1+\|x\|}\ \psi_1(x\ \delta_e) + \psi_1(e_1\ \delta_e)$$

$$= \frac{1}{1+\|x\|}\ y\ \delta_t + e_2\ \delta_{e'}.$$

Therefore $y_1\ \delta_{t_1} = \frac{y}{1+\|x\|}\ \delta_t + e_2\ \delta_{e'}$

It follows that $t = t_1 = e'$ and $y_1 = \frac{y}{1+\|x\|} + e_2$.

Thus we have proved that if x is a positive invertible element of A_1 then there exists a positive invertible element y of A_2 such that $\psi_1(x\ \delta_e) = y\ \delta_{e'}$.

Now suppose that x is a positive element of A_1. We choose $a \in \mathbb{R}^+$ such that $x + ae_1$ is invertible. Then $x + ae_1$ is a positive invertible element of A_1. Therefore there exists a positive invertible element y_1 of A_2 such that

$$\psi_1\left((x+ae_1)\ \delta_e\right) = y_1\ \delta_{e'}.$$

Further,

$$\psi_1\left((x+ae_1)\ \delta_e\right) = \psi_1(x\ \delta_e) + ae_2\ \delta_{e'}$$

Therefore

$$\psi_1(x\ \delta_e) = y_1\ \delta_{e'} - ae_2\ \delta_{e'}$$

$$= (y_1 - ae_2)\ \delta_{e'}.$$

Thus if x is a positive element of A_1, there exists an element y of A_2 such that $\psi_1(x\ \delta_e) = y\ \delta_{e'}$.

Since for each $x \in A_1$, we have

$x = x^+ - x^-$ where x^+, x^- are positive elements of A_1,

we conclude that for each $x \in A_1$ there exists an element y of A_2 such that $\psi_1(x\ \delta_e) = y\ \delta_{e'}$.

We define $F : A_1 \longrightarrow A_2$ by

$$F(x) = y \text{ where } \psi_1(x\ \delta_e) = y\ \delta_{e'}$$

It can be easily seen that F is a continuous positive isomorphism of A_1 onto A_2. Replacing ψ_1 by ψ_1^{-1}, we get that F^{-1} is positive. Hence F is continuous bipositive isomorphism of A_1 onto A_2.

It can be shown exactly as in the isometric case that the mapping ϕ is bijective and the mapping $\beta : G_1 \longrightarrow J^+(A_2)$ is continuous

Remark 6.3.2 : In Theorem 6 3.1, we have considered continuous bipositive isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$. But the arguments of the proof work if we replace $L^1(G_1,A_1)$ by Segal algebra $S(G_1,A_1)$ consisting of vector valued functions whose multiplier space is isomorphic to $M(G_1,A_1)$.

The Banach lattice algebras A_1's in Theorem 6.3.1 satisfy the condition that inverse of every positive invertible element is positive. The examples of Banach lattice algebras given in (1) and (2) below satisfy this condition whereas (3) does not satisfy this condition.

(1) $C^b(G)$: the space of all bounded real valued continuous functions defined on a topological space.

(2) $\ell_\infty(\mathbf{Z})$.

(3) $\ell_1(\mathbf{Z})$. In this case, we find a positive invertible element x of $\ell_1(\mathbf{Z})$ such that x^{-1} is not positive.

Define $x \in \ell_1(\mathbb{Z})$ by

$x_o = 0$

$x_j = \frac{1}{8j^2}$ if $j \neq 0$.

Then $\|x\|_1 = \sum_{-\infty}^{\infty} |x_j| = \sum_{\substack{-\infty \\ j\neq 0}}^{\infty} \frac{1}{8j^2} < 1.$

Let $z = x+e$ where $e = \chi_{\{0\}}$ is the identity of $\ell_1(\mathbb{Z})$.

Then z is positive and invertible. Let y be the inverse of z

Then $z * y = e$

Therefore

$(z * y)(j) = 0$ if $j \neq 0$.

Thus $\sum_k z(j-k)\ y(k) = 0$ if $j \neq 0$

Fix a $j \neq 0$. If $y \geq 0$ then

$z(j-k)\ y(k) = 0$ for each k.

Therefore $y(k) = 0$ for each k which contradicts that y is invertible.

Lastly, in the following Theorem, we give a characterization of continuous bipositive isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$. We omit the proof of this theorem because it is exactly similar to the proof of Theorem 5.4.5. Let λ_1, λ_2 denote the Haar measures on G_1, G_2 respectively.

Theorem 6.3.3 : Let ϕ be a topological isomorphism of G_1 onto G_2, F be a continuous bipositive isomorphism of A_1 onto A_2 and β be a bounded continuous mapping of G_1 into $J^+(A_2)$ with

$\beta(st) = \beta(s)\ \beta(t)$ for $s,t \in G_1$.

Suppose c is the unique positive constant such that $\lambda_1(E) = c\ \lambda_2(\phi(E))$ for all Borel measurable subsets E of G_1. Then ψ defined by

$$(\psi f)(\phi(s)) = c\ \beta(s)\ F(f(s)) \text{ a.e. for } f \in L^1(G_1,A_1)$$

is a continuous bipositive isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ Conversely, every continuous bipositive isomorphism of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ is given in this form.

We have seen in the proof of Theorem 6.3 1 that every continuous bipositive isomorphism ψ of $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$ can be extended to $M(L^1(G_1,A_1)) \cong M(G_1,A_1)$ as a continuous bipositive isomorphism ψ_1 of $M(G_1,A_1)$ onto $M(G_2,A_2)$ defined by $\psi_1(T) = \psi\ T\ \psi^{-1}$ for $T \in M(L^1(G_1,A_1))$. Using arguments similar to that for the proof of Theorem 5.4.4, it can be shown that any continuous isomorphism of $M(G_1,A_1)$ onto $M(G_2,A_2)$ maps $L^1(G_1,A_1)$ onto $L^1(G_2,A_2)$.

Remark 6.3.4: Throughout the Chapter, A, A_1, A_2 are assumed to be Banach lattice algebras. However, we note that the proof of all the results of this Chapter hold even for complex Banach lattice algebras.

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